Mining Human Proofs from Machine Proofs

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Machine Proof	Human Proof
Difficult to "understand"	Focus on understanding "why" theorem is true
All formulas treated the same way	More "interesting" sub-goals highlighted as "Lemmas"
Prove exactly what needs to be proven	Generalising often makes proofs easier/shorter
No new concepts	Special treatment for new "important" ideas/ constructions

Is A provable in L?

formula →

Is A provable in L?

formula → Is A provable in L? ← logic

formula

logic

Is A provable in L?

reduce

formula

logic

Is A provable in L?

reduce

Is E true in C?

formula

logic

Is A provable in L?

reduce

Is E true in C?

equation

formula

logic

Is A provable in L?

reduce

Is E true in C?

equation

class of algebras

formula → Is A provable in L? ← logic

reduce

Is E true in C?

equation → ↻ ← class of algebras

machine finds
equational proof

formula

logic

Is A provable in L?

reduce

recover natural deduction proof

Is E true in C?

equation

class of algebras

machine finds equational proof

Is

$$\neg\neg(\neg\neg\neg A \to A)$$

provable in intuitionistic Lukasiewicz logic?

Does

$$\neg\neg(\neg\neg\neg x \to x) = 0$$

hold in all bounded hoops?

pocrim

$\langle X, \otimes, \to, \geq, 0 \rangle$

partially ordered
commutative
residuated integral
monoid

hoops

Buchi/Owens'74

pocrims satisfying:

If $x \geq y$ then

$$x = y \otimes (y \to x)$$

prover9

- Automated theorem prover for first-order and equational logic
- Successor of Otter
- Developed by Bill McCune
- Uses resolution and paramodulation

http://www.cs.unm.edu/~mccune/prover9/

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```
40 x + (x ==> 1) = 1.  [copy(39),flip(a)].
41 1 ==> x = y ==> ((y ==> 1) ==> x).  [para(40(a,1),5(a,1,1))].
42 x ==> ((x ==> 1) ==> y) = 1 ==> y.  [copy(41),flip(a)].
43 x + 1 = y + (x + (y ==> 1)).  [para(40(a,1),18(a,1,2))].
44 1 = y + (x + (y ==> 1)).  [para(9(a,1),43(a,1))].
45 x + (y + (x ==> 1)) = 1.  [copy(44),flip(a)].
46 x + (y ==> (x ==> z)) = (y ==> z) + ((y ==> z) ==> x).  [para(22(a,1),6(a,1,2))].
47 (x ==> y) + ((x ==> y) ==> z) = z + (x ==> (z ==> y)).  [copy(46),flip(a)].
48 x ==> 0 = y ==> (x ==> y).  [para(7(a,1),22(a,1,2))].
49 0 = y ==> (x ==> y).  [para(8(a,1),48(a,1))].
50 x ==> (y ==> x) = 0.  [copy(49),flip(a)].
51 x ==> 0 = y ==> (x ==> ((y ==> z) ==> z)).  [para(29(a,1),22(a,1,2))].
52 0 = y ==> (x ==> ((y ==> z) ==> z)).  [para(8(a,1),51(a,1))].
53 x ==> (y ==> ((x ==> z) ==> z)) = 0.  [copy(52),flip(a)].
54 1 ==> x = 0.  [para(37(a,1),7(a,1))].
55 x ==> ((x ==> 1) ==> y) = 0.  [para(54(a,1),42(a,2))].
56 1 = x + ((x ==> y) + (y ==> 1)).  [para(45(a,1),24(a,1))].
57 x + ((x ==> y) + (y ==> 1)) = 1.  [copy(56),flip(a)].
58 x ==> (0 ==> y) = (z ==> x) ==> (((z ==> x) ==> x) ==> y).  [para(50(a,1),26(a,1,2,1))].
59 x ==> y = (z ==> x) ==> (((z ==> x) ==> x) ==> y).  [para(33(a,1),58(a,1,2))].
60 (x ==> y) ==> (((x ==> y) ==> y) ==> z) = y ==> z.  [copy(59),flip(a)].
61 x ==> ((x ==> y) ==> ((y ==> z) ==> z)) = 0.  [para(26(a,1),53(a,1))].
62 x + (0 + (((x ==> y) ==> y) ==> 1)) = 1.  [para(29(a,1),57(a,1,2,1))].
63 x + (((x ==> y) ==> y) ==> 1) = 1.  [para(20(a,1),62(a,1,2))].
64 x + ((y ==> z) + ((y ==> z) ==> u)) = u + (x + (y ==> (u ==> z))).  [para(22(a,1),38(a,2,2,2))].
65 1 ==> x = y ==> ((((y ==> z) ==> z) ==> 1) ==> x).  [para(63(a,1),5(a,1,1))].
66 0 = y ==> ((((y ==> z) ==> z) ==> 1) ==> x).  [para(54(a,1),65(a,1))].
67 x ==> ((((x ==> y) ==> y) ==> 1) ==> z) = 0.  [copy(66),flip(a)].
68 (x ==> y) + ((x ==> y) ==> (x ==> 1)) = (x ==> 1) + 0.  [para(55(a,1),47(a,2,2))].
69 (x ==> y) + (x ==> ((x ==> y) ==> 1)) = (x ==> 1) + 0.  [para(22(a,1),68(a,1,2))].
70 (x ==> y) + (x ==> ((x ==> y) ==> 1)) = 0 + (x ==> 1).  [para(3(a,1),69(a,2))].
71 (x ==> y) + (x ==> ((x ==> y) ==> 1)) = x ==> 1.  [para(20(a,1),70(a,2))].
```

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- But we started noticing some patterns...

Certain derived connectives kept appearing:

weak conjunction

$$A \wedge B \equiv A \otimes (A \to B)$$

strong disjunction

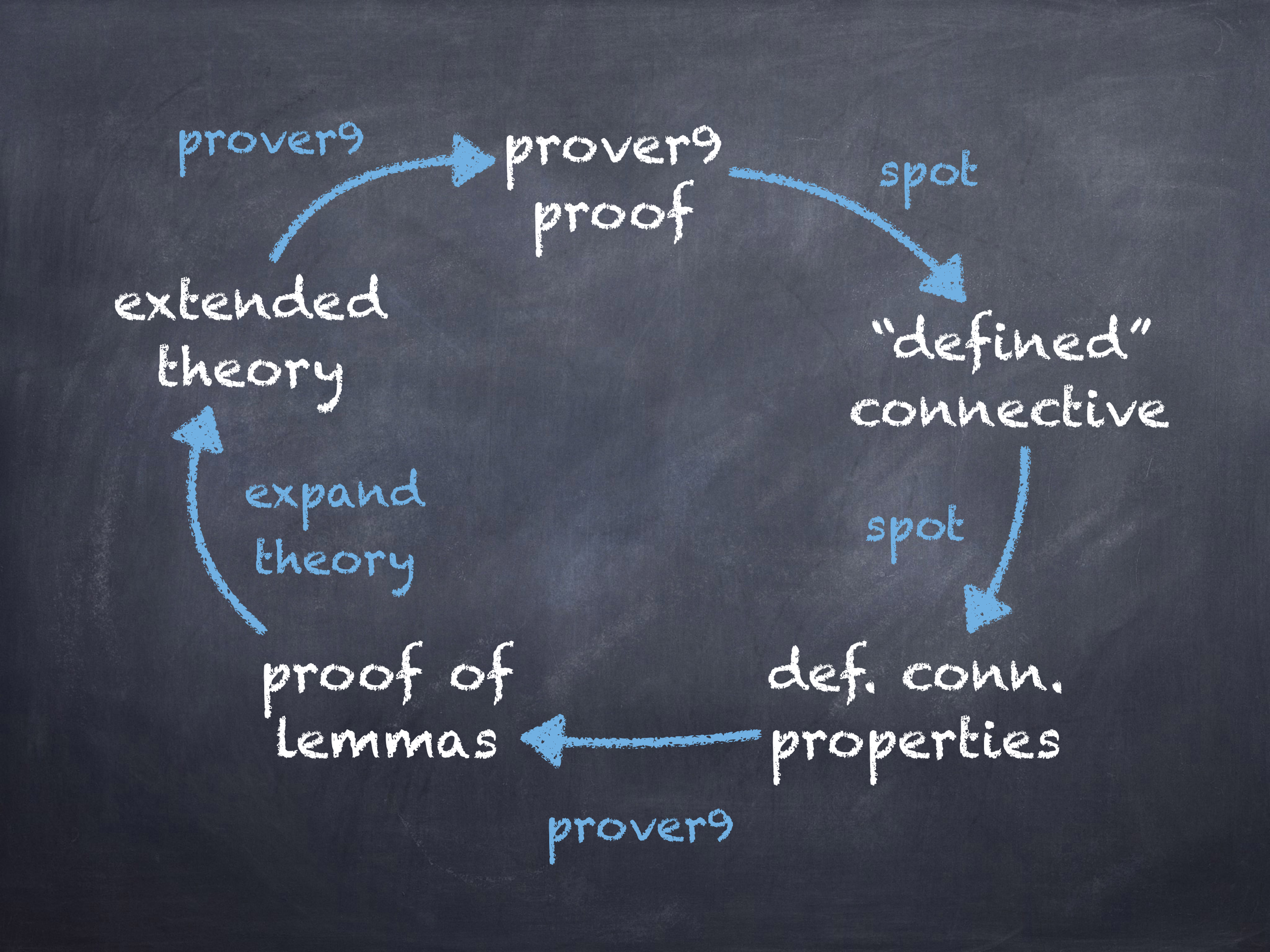
$$A \vee B \equiv (B \to A) \to A$$

strong implication

$$A \Rightarrow B \equiv A \to A \otimes B$$

NOR, Peirce's ampheck

$$A \downarrow B \equiv \neg A \otimes (B \to A)$$


prover9
prover9 proof
spot
extended theory
"defined" connective
expand theory
spot
proof of lemmas
def. conn. properties
prover9

Lemma 4.2 ($\mathbf{\text{ŁL}_i}$) $A \otimes B \;\leftrightarrow\; A \otimes (B \vee (A \Rightarrow B))$

Theorem 4.7 ($\mathbf{\text{ŁL}_i}$) $B \downarrow A \;\leftrightarrow\; A \downarrow B$

Corollary 4.8 ($\mathbf{\text{ŁL}_i}$) $(A^{\perp\perp} \multimap A)^{\perp\perp}$

Proof: Note that, since $\perp \leftrightarrow\; A \otimes A^{\perp}$ we have $(*)$ $A^{\perp\perp} \;\leftrightarrow\; A^{\perp} \Rightarrow A$. Moreover, it is easy to check that $(**)$ $X \downarrow (Y \multimap X) \;\leftrightarrow\; X^{\perp} \otimes (X \vee Y)$, for all X and Y. Hence

$$
\begin{aligned}
(A^{\perp\perp} \multimap A)^{\perp} &\leftrightarrow ((A^{\perp} \Rightarrow A) \multimap A)^{\perp} && (*)\\
&\leftrightarrow ((A^{\perp} \Rightarrow A) \multimap A)^{\perp} \otimes \underline{(A \multimap ((A^{\perp} \Rightarrow A) \multimap A))} && ([\mathsf{WK}])\\
&\leftrightarrow ((A^{\perp} \Rightarrow A) \multimap A) \downarrow A && (\text{def } \downarrow)\\
&\leftrightarrow A \downarrow ((A^{\perp} \Rightarrow A) \multimap A) && (\text{Theorem } 4.7)\\
&\leftrightarrow A^{\perp} \otimes (A \vee (A^{\perp} \Rightarrow A)) && (**)\\
&\leftrightarrow A^{\perp} \otimes A && (\text{Lemma } 4.2)\\
&\leftrightarrow \perp . && \blacksquare
\end{aligned}
$$

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- Human input is identifying the "right" abstractions:
  - Find useful derived concepts
  - Recover an intuitive proof plan
- Automated support for proof refactoring?
- AI to automate human aspect?
- The late Bill McCune is the real star!