Kripke Semantics for Intuitionistic Łukasiewicz Logic

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Double Negation Semantics for Generalisations of Heyting Algebras

(both published in Studia Logica)

1. Intuitionistic Łukasiewicz Logic

2. Algebraic Semantics

3. Kripke Semantics

1. Logics

Intuitionistic Affine Logic

$$\frac{}{\Gamma, A \vdash A} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A_1 \otimes A_2}{\Gamma \vdash A_i} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}$$

$$\frac{}{\Gamma, x\colon A \vdash x\colon A} \qquad \frac{\Gamma \vdash s\colon A \quad \Delta \vdash t\colon B}{\Gamma, \Delta \vdash \langle s, t\rangle\colon A \otimes B} \qquad \frac{\Gamma, x\colon A \vdash s\colon B}{\Gamma \vdash \lambda x\,.\,s\colon A \to B}$$

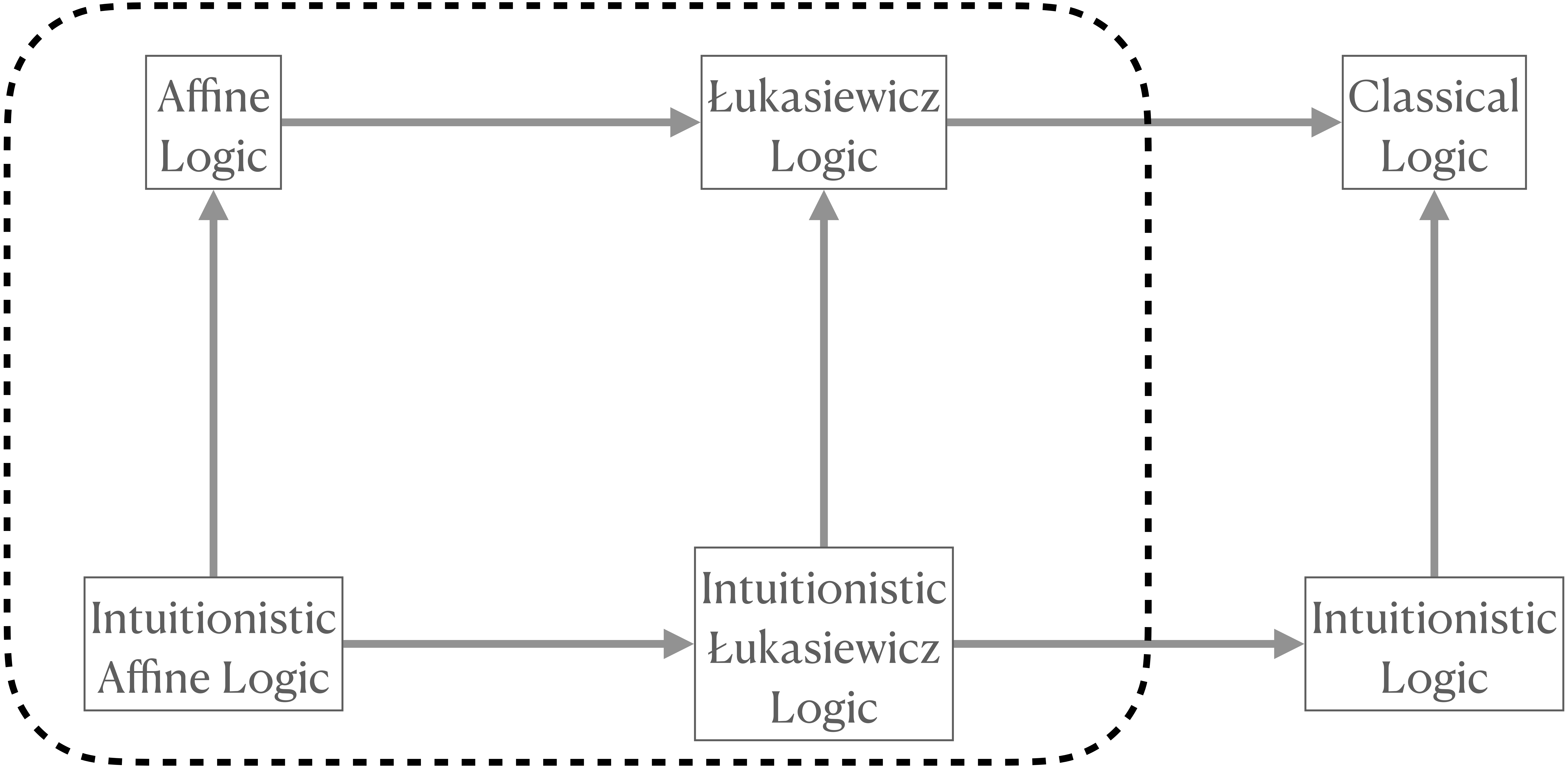
$$\frac{\Gamma \vdash s\colon \bot}{\Gamma \vdash \mathscr{A}(s)\colon A} \qquad \frac{\Gamma \vdash s\colon A_1 \otimes A_2}{\Gamma \vdash \pi_i(s)\colon A_i} \qquad \frac{\Gamma \vdash t\colon A \quad \Delta \vdash s\colon A \to B}{\Gamma, \Delta \vdash st\colon B}$$

Intuitionistic
Affine Logic


Affine
Logic
Intuitionistic
Affine Logic
Intuitionistic
Logic


Affine
Logic
Classical
Logic
Intuitionistic
Affine Logic
Intuitionistic
Logic

Sub-structural logics = lacking some structural rules



Structural Rules

Exchange (E): $$\frac{\Gamma, A, B, \Delta \vdash B}{\Gamma, B, A, \Delta \vdash B}$$

Weakening (W): $$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

Contraction (C): $$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

Involution of Negation

(I): $$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

{E, W, C, I}: Classical Logic

{ }: Lambek Calculus

(non-commutative LL)

Here we assume...

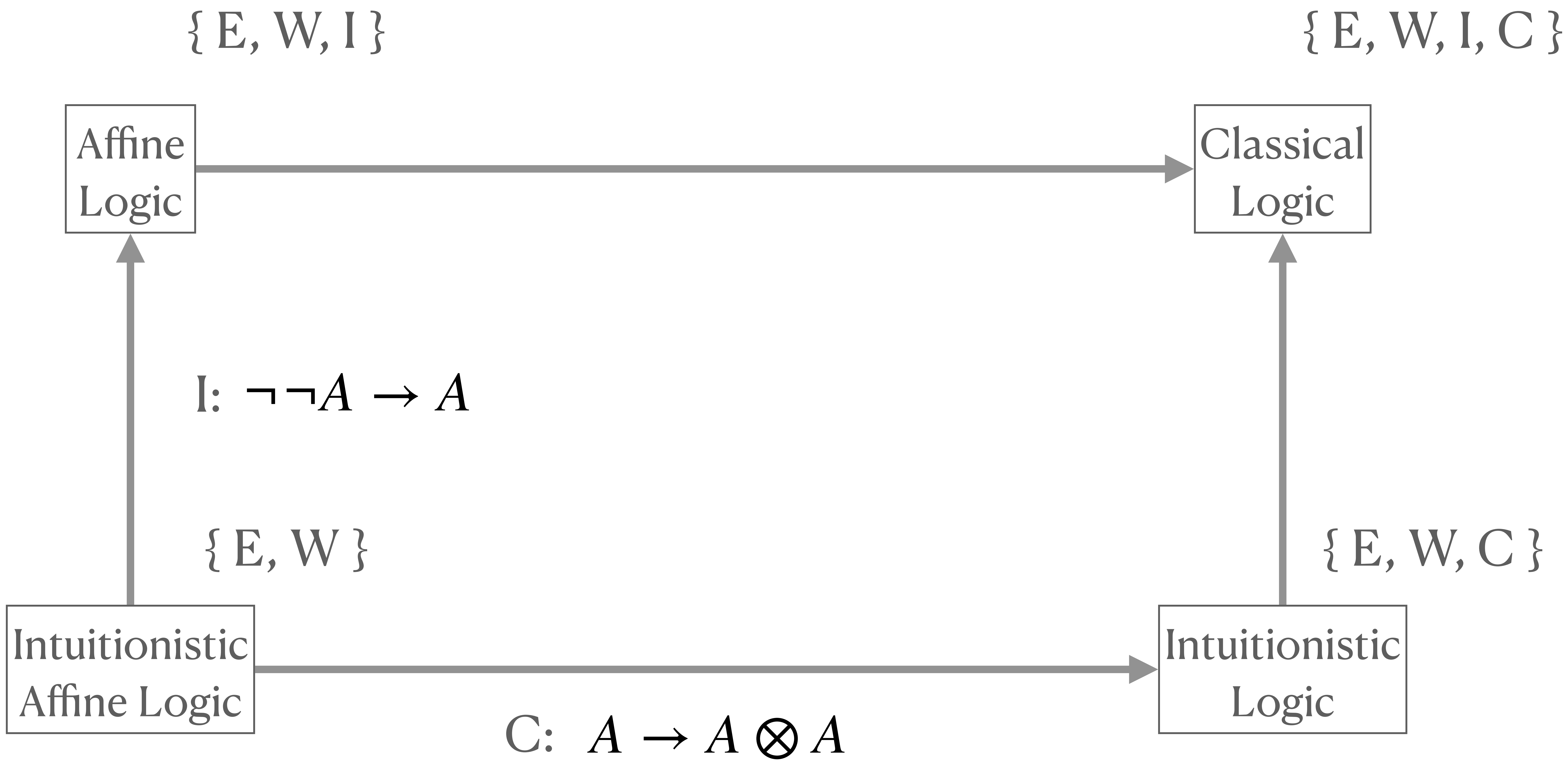
(E): $$\frac{\Gamma, A, B, \Delta \vdash B}{\Gamma, B, A, \Delta \vdash B}$$

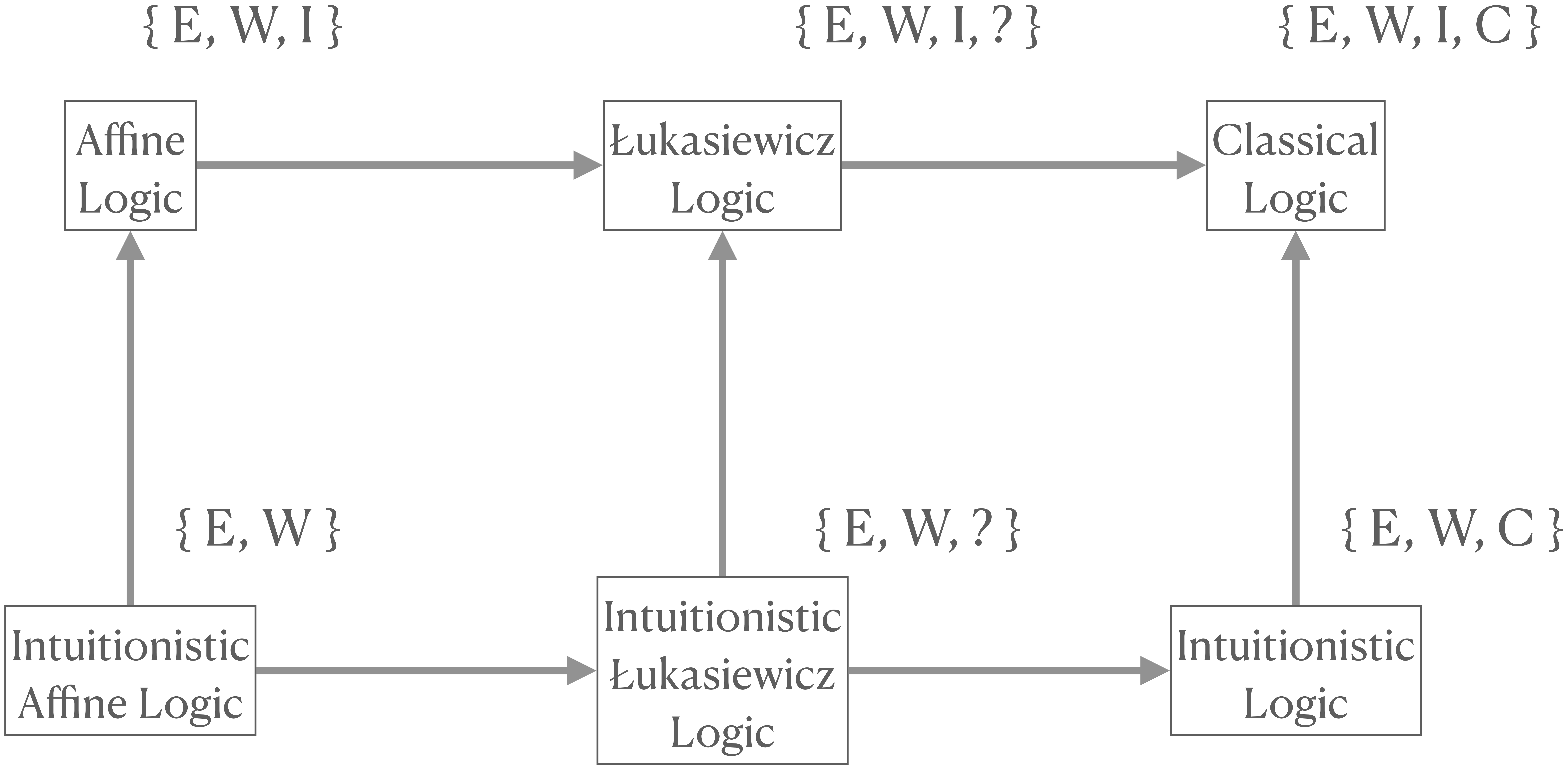
(W): $$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

... and play with...

(I): $$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

(C): $$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$


{ E, W, I }
{ E, W, I, C }
Affine Logic
Classical Logic
I: $\neg\neg A \to A$
{ E, W }
{ E, W, C }
Intuitionistic Affine Logic
Intuitionistic Logic
C: $A \to A \otimes A$


{ E, W, I }
{ E, W, I, ? }
{ E, W, I, C }
Affine Logic
Łukasiewicz Logic
Classical Logic
{ E, W }
{ E, W, ? }
{ E, W, C }
Intuitionistic Affine Logic
Intuitionistic Łukasiewicz Logic
Intuitionistic Logic

A weak form of contraction...

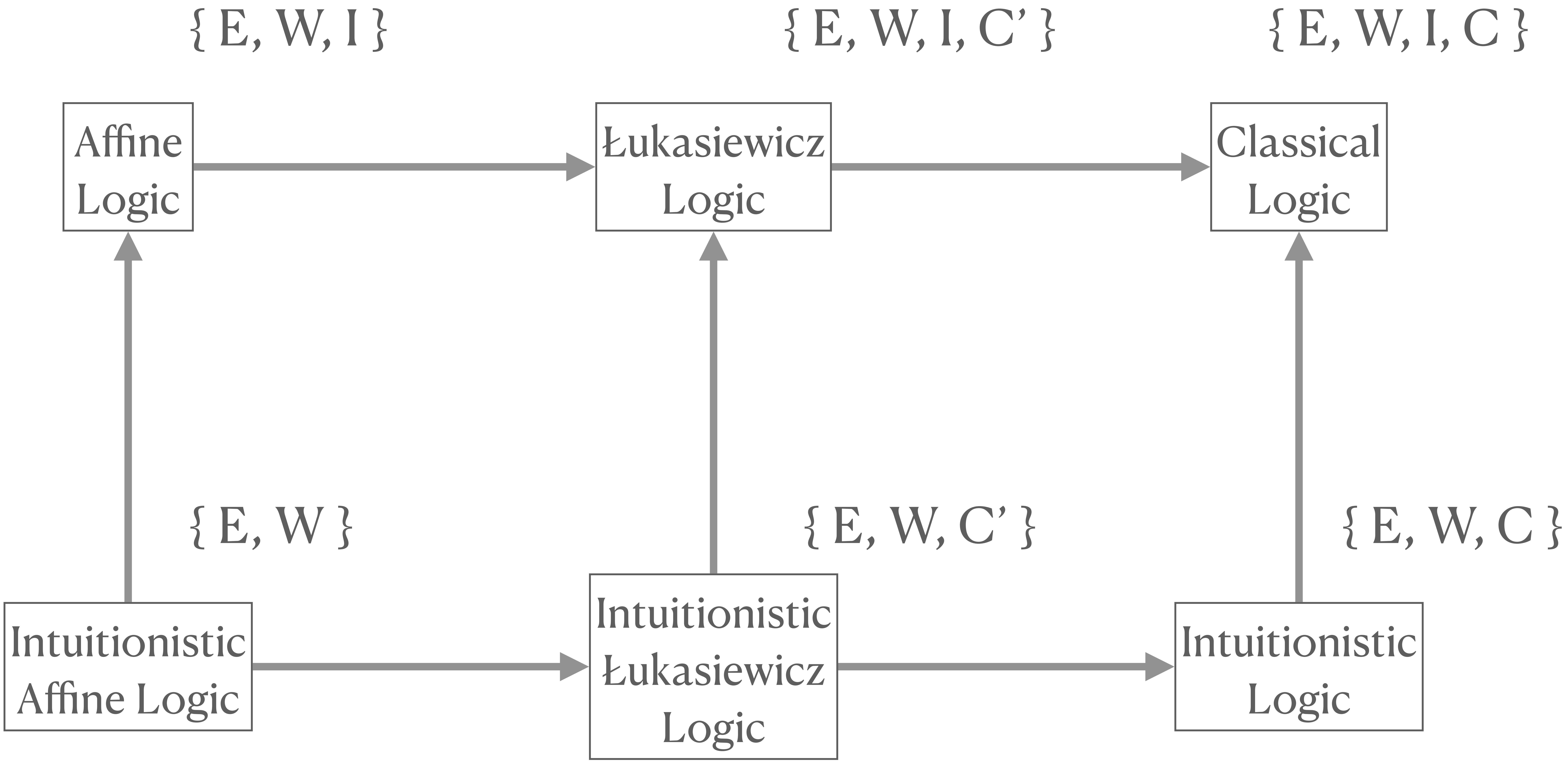
(C'): $$\frac{\Gamma, (B \to A) \to A, B \to A \vdash C}{\Gamma, A \vdash C}$$

compare to full contraction

$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

equivalent to the axiom...

$$\frac{}{A, A \to B \vdash B \otimes (B \to A)}$$


{ E, W, I }
{ E, W, I, C' }
{ E, W, I, C }
Affine Logic
Łukasiewicz Logic
Classical Logic
{ E, W }
{ E, W, C' }
{ E, W, C }
Intuitionistic Affine Logic
Intuitionistic Łukasiewicz Logic
Intuitionistic Logic

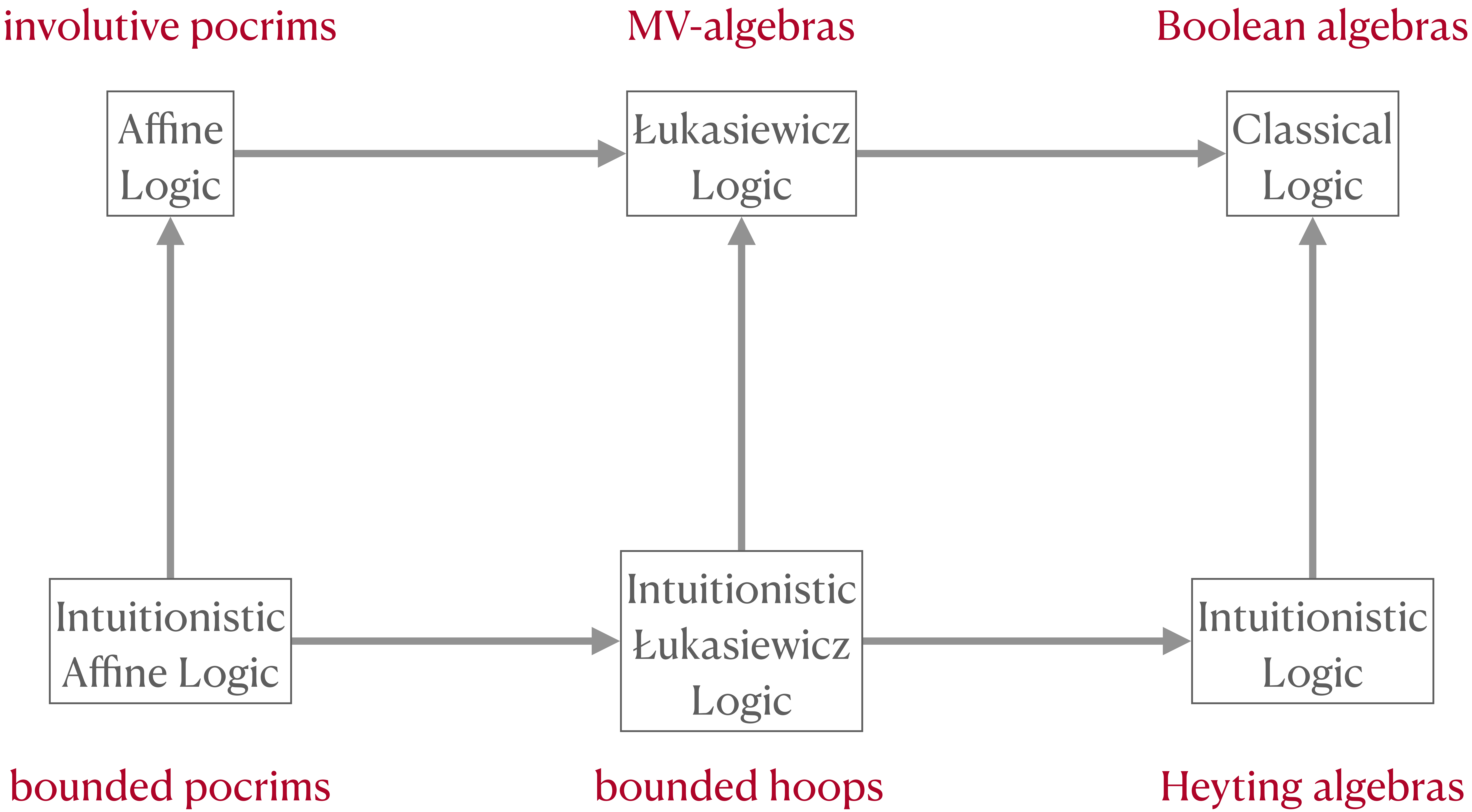
“Translations of classical into intuitionistic sub-structural logics”

$$
\begin{tikzcd}
\text{Affine Logic} \arrow[r] \arrow[d, dashed, bend right] & \text{Łukasiewicz Logic} \arrow[r] \arrow[d, dashed, bend right] & \text{Classical Logic} \arrow[d, dashed, bend right] \\
\text{Intuitionistic Affine Logic} \arrow[u] \arrow[r] & \text{Intuitionistic Łukasiewicz Logic} \arrow[u] \arrow[r] & \text{Intuitionistic Logic} \arrow[u]
\end{tikzcd}
$$

“A Kripke Semantics for Intuitionistic Lukasiewicz Logic”

$$
\begin{tikzcd}
\text{Affine Logic} \arrow[r] & \text{Łukasiewicz Logic} \arrow[r] & \text{Classical Logic} \\
\text{Intuitionistic Affine Logic} \arrow[u] \arrow[r] & \boxed{\text{Intuitionistic Łukasiewicz Logic}} \arrow[u] \arrow[r] & \text{Intuitionistic Logic} \arrow[u]
\end{tikzcd}
$$

2. Algebraic Semantics


involutive pocrims
MV-algebras
Boolean algebras
Affine Logic
Łukasiewicz Logic
Classical Logic
Intuitionistic Affine Logic
Intuitionistic Łukasiewicz Logic
Intuitionistic Logic
bounded pocrims
bounded hoops
Heyting algebras

bounded pocrims $\mathbb{P} = \langle P, \cdot\, , \rightarrow\, , 0, 1, \leq \rangle$

bounded

$0 \leq x$

integral

$x \leq 1$

partial order

$x \leq x$

$x \leq y, y \leq z \Rightarrow x \leq z$

$x \leq y, y \leq x \Rightarrow x = y$

residuated

$x \cdot y \leq z \;\Leftrightarrow\; x \leq y \rightarrow z$

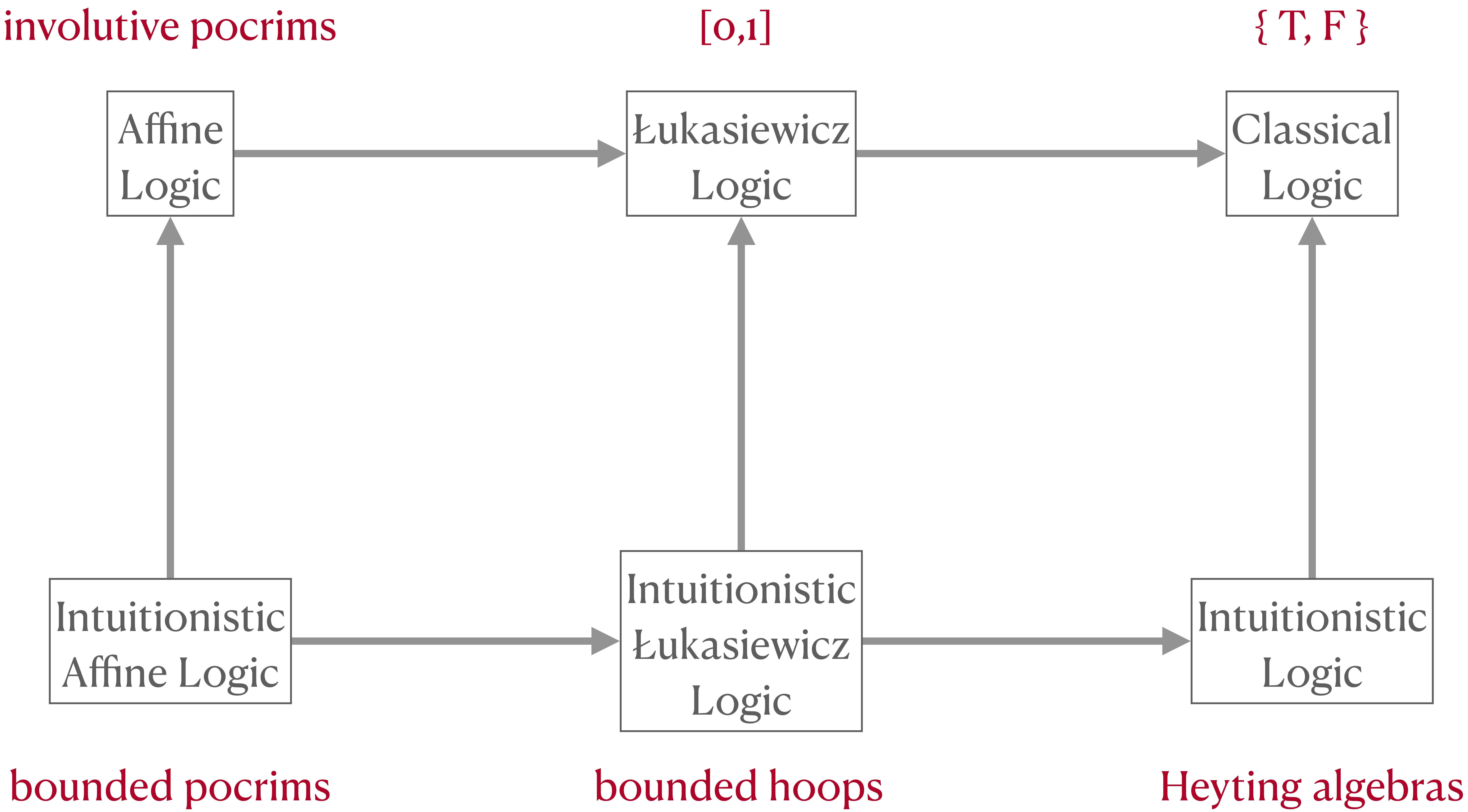
commutative

$x \cdot y = y \cdot x$

monoid

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x \cdot 1 = x$

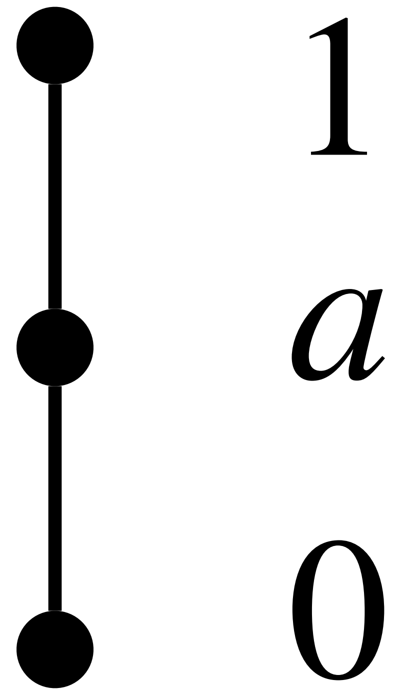

involutive pocrims
[0,1]
{ T, F }
Affine Logic
Łukasiewicz Logic
Classical Logic
Intuitionistic Affine Logic
Intuitionistic Łukasiewicz Logic
Intuitionistic Logic
bounded pocrims
bounded hoops
Heyting algebras

Definition 2.7 (Standard MV-chain). For $x \in [0,1]$, let $\overline{x} := 1 - x$. The *standard MV-chain*, denoted $[0,1]_{\mathrm{MV}}$, is the MV algebra defined as follows: The domain of $[0,1]_{\mathrm{MV}}$ is the unit interval $[0,1]$, with the constants and binary operations defined as

$$
\begin{aligned}
\top &:= 1 \\
\bot &:= 0 \\
x \wedge y &:= \min\{x, y\} \\
x \vee y &:= \max\{x, y\} \\
x \otimes y &:= \max\{0, \overline{\overline{x} + \overline{y}}\} \\
x \to y &:= \min\{1, \overline{\overline{y} - \overline{x}}\}
\end{aligned}
$$

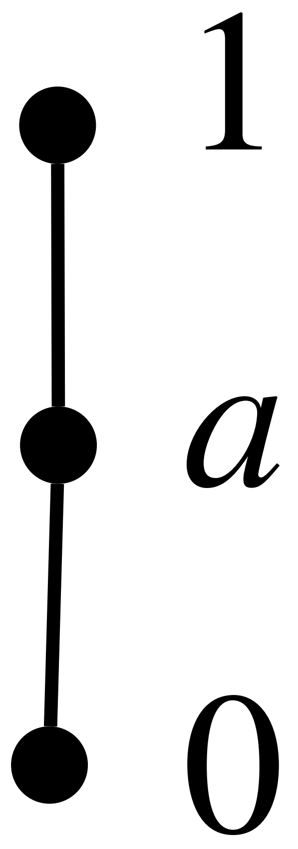
Note 1. $x \otimes y$ is equivalent to $\max\{0, x + y - 1\}$, and $x \to y$ is equivalent to $\min\{1, y - x + 1\}$.

Bounded pocrim where idempotence fails $(x \cdot x \neq x)$



$$a \cdot a = 1$$

Bounded pocrim where involution fails $(x^{\perp\perp} \neq x)$



$$a \cdot a = a$$

$$a^{\perp} = 0$$

$$a^{\perp\perp} = 1$$

3. Kripke Semantics for IŁL

Kripke structure

- partial order $\langle W, \leq \rangle$
- valuation $w \Vdash p \in \mathbb{B}$
- such that $w \nVdash \bot$ and $w \leq v$ and $w \Vdash p$ implies $v \Vdash p$

Kripke semantics
$$w \Vdash A \wedge B = (w \Vdash A) \wedge (w \Vdash B)$$
$$w \Vdash A \rightarrow B = \forall v \geq w((v \Vdash A) \rightarrow (v \Vdash B))$$

<u>Main result</u>: *A* is provable in IL iff *A* is valid in all Kripke structures

Kripke structure (KS)

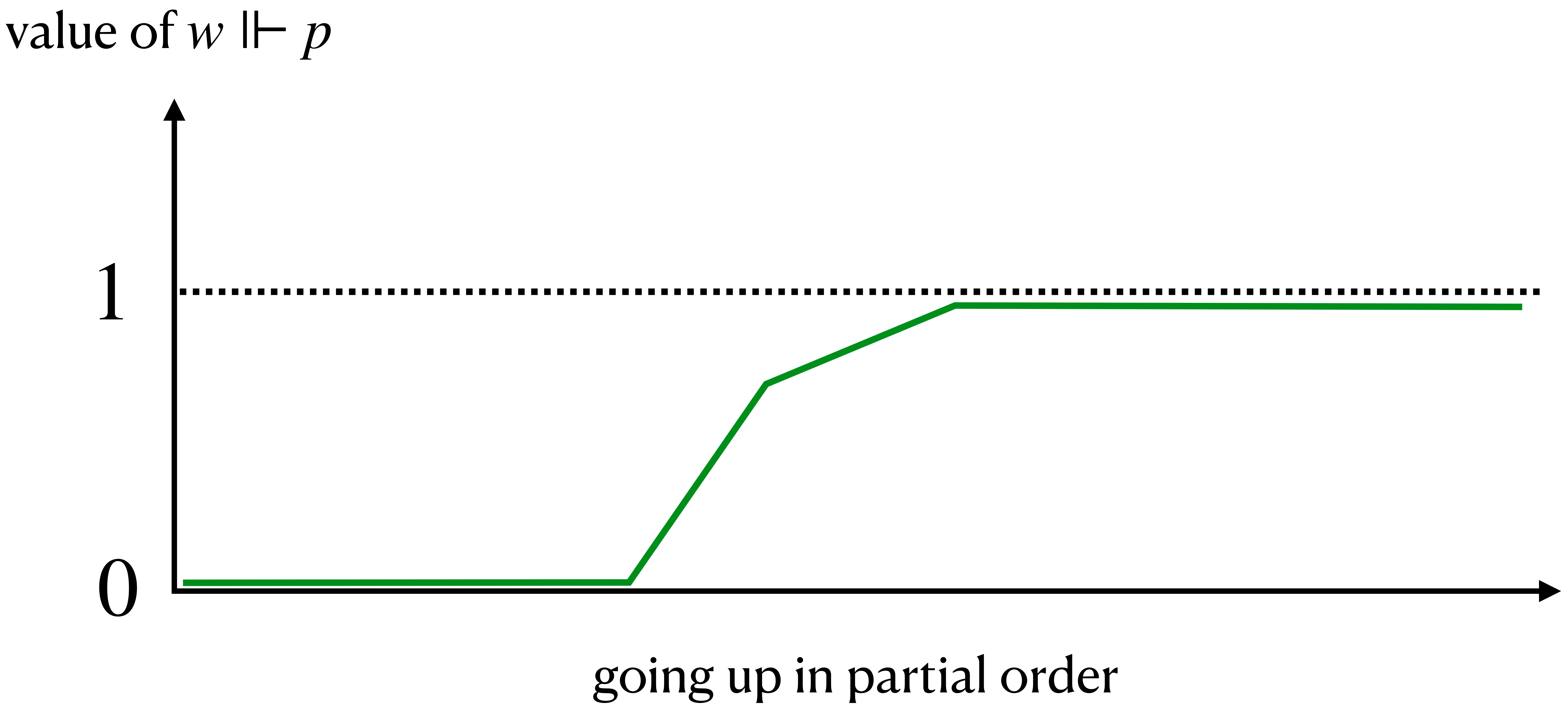
- partial order $\langle W, \leq \rangle$
- valuation $w \Vdash p \in \mathbb{B}$
- such that $w \nVdash \bot$ and $w \leq v$ and $w \Vdash p$ implies $v \Vdash p$

Generalising KS (we call these Bova-Montagna structures)

- partial order $\langle W, \leq \rangle$
- valuation $w \Vdash p \in [0,1]$
- such that $w \Vdash \bot = 0$ and $w \leq v$ and $(w \Vdash p) \leq (v \Vdash p)$

  $w \Vdash p > 0$ and $v > w$ then $v \Vdash p = 1$

$\boxed{\text{We call these “sloping functions”}}$


value of $w \Vdash p$
1
0
going up in partial order

Bova-Montagna structure

We call these "sloping functions"

- partial order $\langle W, \leq \rangle$
- valuation $w \Vdash p \in [0,1]$
- such that $w \Vdash \bot = \bot$ and $w \leq v$ and $(w \Vdash p) \leq (v \Vdash p)$

$w \Vdash p > \bot$ and $v > w$ then $v \Vdash p = \top$

Semantics

$$w \Vdash A \otimes B = (w \Vdash A) \otimes (w \Vdash B)$$

$$w \Vdash A \to B = \lfloor \inf_{v \geq w} \rfloor ((v \Vdash A) \to (v \Vdash B))$$

<u>Main result</u>: A is provable in IŁL iff A is valid in all BM structures

Logical symbol

$$w \Vdash A \otimes B = (w \Vdash A) \otimes (w \Vdash B)$$

Operation in standard MV-chain

$$w \Vdash A \to B = \lfloor \inf_{v \geq w} \rfloor ((v \Vdash A) \to (v \Vdash B))$$

$$\lfloor \mathbf{inf} \rfloor_{v \succeq w} f(v) := \begin{cases} f(w) & \textbf{if}\ \forall v \succ w (f(v) = \top) \\ \bot & \textbf{if}\ \exists v \succ w (f(v) < \top) \end{cases}$$

Main result: A is provable in IŁL iff A is valid in all BM structures

BM structure where
DNE fails

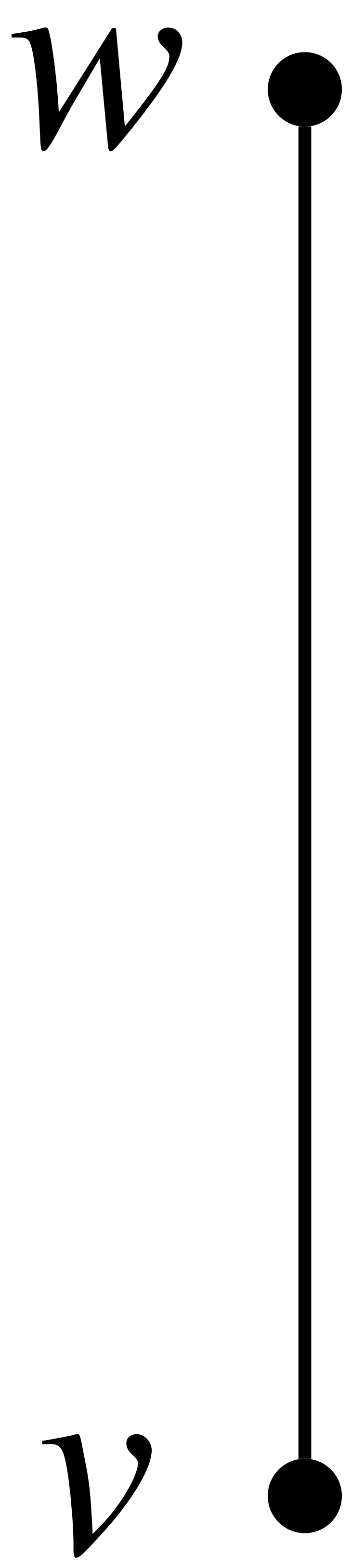
$(p^{\perp\perp} \to p)$

$$v \Vdash p^{\perp\perp} \to p$$

$$\inf_{v' \geq v} (v' \Vdash p^{\perp\perp} \to v' \Vdash p)$$

$$v \Vdash p^{\perp\perp} \to v \Vdash p$$

$$0$$



$w \Vdash p = 1$

$w \Vdash p^{\perp} = 0$

$w \Vdash p^{\perp\perp} = 1$

$v \Vdash p = 0$

$v \Vdash p^{\perp} = 0$

$v \Vdash p^{\perp\perp} = 1$

BM structure where contraction fails

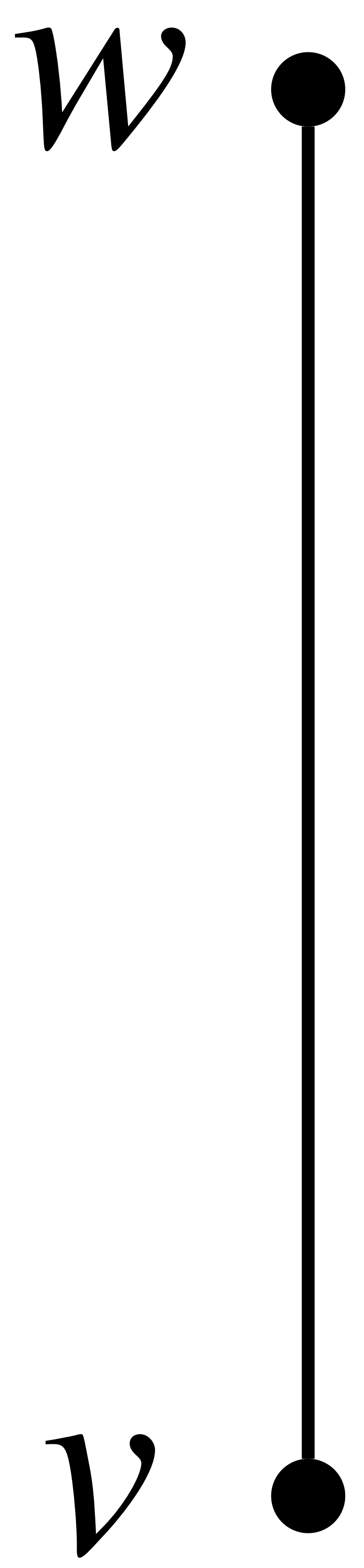
$(p \to p \otimes p)$

$v \Vdash p \to p \otimes p$

$$\inf_{v' \geq v} (v' \Vdash p \to v' \Vdash p \otimes p)$$

$(w \Vdash p) \to (w \Vdash p \otimes p)$

0.5



$w \Vdash p = 0.5$

$w \Vdash p \otimes p = 0$

$(w \Vdash p) \to (w \Vdash p \otimes p) = 0.5$

$v \Vdash p = 0$

$v \Vdash p \otimes p = 0$

$(v \Vdash p) \to (v \Vdash p \otimes p) = 1$

The consequence relation in the logic of commutative *GBL*-algebras is **PSPACE**-complete

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Definition 2 (*Poset Sum*). Let $\mathbf{P} = (P, E_P)$ be the cover graph of a poset $(P, \leq_P)$ and let $(\mathbf{C}_p)_{p \in P}$ be a sequence of standard *MV*-chains. The *(dual) poset sum* $\mathbf{A}$ over the *skeleton* $\mathbf{P}$ and the *summands* $(\mathbf{C}_p)_{p \in P}$, in symbols $\mathbf{A} = \bigoplus_{p \in P} \mathbf{C}_p$, is the algebra of signature $\mathcal{L}$ defined as follows (if $\circ \in \mathcal{L}$, then $\circ_p$ and $\circ_A$ are respectively for the realizations in $\mathbf{C}_p$ and $\mathbf{A}$ of the symbol $\circ$):

- (i) The domain, A, of $\mathbf{A}$ is the set of all maps h on P such that:
  - (i.i) for all $p \in P$, $h(p) \in \mathbf{C}_p$;
  - (i.ii) for all $p \in P$, if $h(p) < \top_p$, then $\bot_q = h(q)$ for all $q \in P$ such that $q <_P p$, and (thus), if $\bot_p < h(p)$, then $h(q) = \top_q$ for all $q \in P$ such that $q >_P p$.
- (ii) The realization of $\mathcal{L}$ in $\mathbf{A}$ is the following. For every $p \in P$ and $h_1, h_2 \in A$:
  - (ii.i) $\bot_A(p) = \bot_p$;
  - (ii.ii) $\top_A(p) = \top_p$;
  - (ii.iii) $(h_1 \odot_A h_2)(p) = h_1(p) \odot_p h_2(p)$;

The consequence relation in the logic of commutative *GBL*-algebras is **PSPACE**-complete

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(ii.iv) $(h_1 \vee_A h_2)(p) = h_1(p) \vee_p h_2(p)$;

(ii.v) $(h_1 \wedge_A h_2)(p) = h_1(p) \wedge_p h_2(p)$;

(ii.vi) The realization of $\rightarrow$ in **A** is the following:

- (ii.vi.i) $(h_1 \rightarrow_A h_2)(p) = h_1(p) \rightarrow_p h_2(p)$, if $h_1(q) \leq h_2(q)$ for all $q \in P$ such that $p <_P q$;
- (ii.vi.ii) $(h_1 \rightarrow_A h_2)(p) = \perp_p$, otherwise;

(ii.vii) $h_1 =_A h_2$ if and only if $h_1(p) = h_2(p)$ for all $p \in P$;

(ii.viii) $h_1 \leq_A h_2$ if and only if $h_1(p) \leq h_2(p)$ for all $p \in P$;

(ii.ix) $h_1 <_A h_2$ if and only $h_1 \leq_A h_2$ and $h_1(p) < h_2(p)$ for some $p \in P$.

Theorem 1 (*Jipsen and Montagna*). *Let E be a quasiequation. Then, $\langle E \rangle \notin$ GBL-CB-QEQ if and only if there exists a finite poset sum **A** such that **A** $\not\models E$.*