On the struct[illegible] of realizability and functio[illegible]terpretations

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My Years under Ulrich's PhD Supervision

... and how it shaped all the research I've done in these past 20 years

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Monday 29 August 2022

(*dedicated to Ulrich Kohlenbach on his 60th birthday*)

Logic

Incompleteness
vs
Completeness

Mathematical Logic

Joseph R. Shoenfield

Proof Mining

2001

Proof mining in L_1-approximation

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Abstract

In this paper we present another case study in the general project of proof mining which means the logical analysis of prima facie non-effective proofs with the aim of extracting new computationally relevant data. We use techniques based on monotone functional interpretation (developed in [17]) to analyze Cheney's simplification [6] of Jackson's original proof [10] from 1921 of the uniqueness of the best L_1-approximation of continuous functions $f \in C[0,1]$ by polynomials $p \in P_n$ of degree $\leq n$. Cheney's proof is non-effective in the sense that it is based on classical logic and on the non-computational principle **WKL** (binary König's lemma). The result of our analysis provides the first effective (in all parameters) uniform

Lemma 3.1 ([7], Lemma 1) *Let* $f, h \in C[0,1]$*. If* f *has at most finitely many roots and if* $\int_0^1 h\, sgn(f) \neq 0$*, then for some* $\lambda \in \mathbb{R}$*,* $\int_0^1 |f - \lambda h| < \int_0^1 |f|$*,*

where

$$sgn(f)(x) \stackrel{\mathbb{N}}{=} \begin{cases} 1, \ \textit{if} \ f(x) >_{\mathbb{R}} 0 \\ 0, \ \textit{if} \ f(x) =_{\mathbb{R}} 0 \\ -1, \ \textit{if} \ f(x) <_{\mathbb{R}} 0. \end{cases}$$

Main result (Theorem 4.1) *Let* $\Phi(\omega, n, \varepsilon) :\equiv \min\{\frac{c_n \varepsilon}{8(n+1)^2}, \frac{c_n \varepsilon}{2}\omega_{f,n}(\frac{c_n\varepsilon}{2})\}$, where

$$c_n :\equiv \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{2^{4n+3}(n+1)^{3n+1}} \text{ and } \omega_n(\varepsilon) :\equiv \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 \lceil \frac{1}{\omega(1)} \rceil}\}.$$

The functional Φ *is a uniform modulus of uniqueness for the best* L_1*-approximation of any function* f *in* $C[0,1]$ *having modulus of uniform continuity* ω *from* P_n*, i.e.*

$$\begin{cases} \forall n \in \mathbb{N}; p_1, p_2 \in P_n; \varepsilon \in \mathbb{Q}^*_+ \\ \quad \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) < \Phi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \leq \varepsilon \Big), \end{cases}$$

where $dist_1(f, P_n) :\equiv \inf_{p \in P_n} \|f - p\|_1$ *and* $\omega : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ *is a modulus of uniform continuity for* $f \in C[0,1]$ *if* [5]

$$\forall x, y \in [0,1]; \varepsilon \in \mathbb{Q}^*_+ (|x - y| < \omega(\varepsilon) \to |f(x) - f(y)| < \varepsilon).$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A] \qquad \dfrac{[A] \qquad A \rightarrow B}{B}}{A \wedge B} \qquad A \wedge B \rightarrow C}{C} \qquad C_1}{C \wedge C_1} \qquad C \wedge C_1 \rightarrow D}{D} \qquad D \rightarrow K}{K}
$$

2002

PROOF MINING: A SYSTEMATIC WAY OF ANALYSING PROOFS IN MATHEMATICS

ULRICH KOHLENBACH AND PAULO OLIVA

Abstract. We call *proof mining* the process of logically analyzing proofs in mathematics with the aim of obtaining new information. In this survey paper we discuss, by means of examples from mathematics, some of the main techniques used in proof mining. We show that those techniques not only apply to proofs based on classical logic, but also to proofs which involve non-effective principles such as the attainment of the infimum of $f \in C[0,1]$ and the convergence for bounded monotone sequences of reals. We also report on recent case studies in approximation theory and fixed point theory where new results were obtained.

$$\forall x \in X \forall y \in K \; (f(x,y) = 0 \to g(x,y) = 0). \tag{2}$$

4.1. Uniqueness. Let (X, d_X) and (K, d_K) be Polish spaces, K compact. The fact that a $\mathcal{T}^\omega$-definable (and hence continuous) function $f : X \times K \to \mathbb{R}$ for each given $x \in X$ has at most one root in K can be expressed as

$$\mathsf{UNI}(f) :\equiv \forall x \in X; y_1, y_2 \in K(\bigwedge_{i=1}^{2} f(x, y_i) \stackrel{\mathbb{R}}{=} 0 \to d_K(y_1, y_2) \stackrel{\mathbb{R}}{=} 0),$$

which has the form (2). The *monotone functional interpretation* of a uniqueness statement of the form UNI creates a modulus $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ such that

$$\forall x \in X; y_1, y_2 \in K; \varepsilon \in \mathbb{Q}^*_+(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon),$$

4.2. Convexity. Let $(X, \|\cdot\|)$ denote a normed linear space whose unit ball $B :\equiv \{x \in X : \|x\| \leq 1\}$ is compact (which – classically – amounts to X being finite dimensional). From the statement that X is strictly convex

$$\mathsf{CVX} :\equiv \forall x, y \in B(\|\frac{1}{2}(x+y)\| \overset{\mathbb{R}}{=} 1 \rightarrow \|x-y\| \overset{\mathbb{R}}{=} 0),$$

which is again of the form (2), *monotone functional interpretation* creates a modulus $\eta : \mathbb{Q}^*_+ \rightarrow \mathbb{Q}^*_+$ satisfying

$$\forall x, y \in B; \varepsilon \in \mathbb{Q}^*_+(\|\frac{1}{2}(x+y)\| > 1 - \eta(\varepsilon) \rightarrow \|x-y\| < \varepsilon).$$

$$(1) \qquad \forall x \in X \forall y \in K \; (f(x,y) > 0 \to g(x,y) > 0),$$

4.3. Contractivity. Let (K,d) be a compact Polish space. A function $f : K \to K$ is defined to be *contractive* if[12]

$$\mathsf{CTR}(f) :\equiv \forall x, y \in K (x \neq y \to d(f(x), f(y)) < d(x,y)),$$

which has the form (1). The *monotone functional interpretation* of the statement that a $\mathcal{T}^\omega$-definable f is contractive creates a modulus $\eta : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ satisfying

$$\forall x, y \in K; \varepsilon \in \mathbb{Q}^*_+ (d(x,y) > \varepsilon \to d(f(x), f(y)) + \eta(\varepsilon) < d(x,y)).$$

4.5. Monotonicity. Let $f : [0,1] \to \mathbb{R}$ be a $\mathcal{T}^\omega$-definable strictly increasing (decreasing) function, i.e.,

$$\mathsf{MON}(f) :\equiv \forall x, y \in [0,1](x - y > 0 \to f(x) - f(y) > 0),$$

which has the form (1). From this statement *monotone functional interpretation* creates a modulus $\delta : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ such that

$$\forall x, y \in [0,1]; \varepsilon \in \mathbb{Q}^*_+ (x - y > \varepsilon \to f(x) - f(y) > \delta(\varepsilon)),$$

called *modulus of monotonicity*. Note that the modulus of monotonicity δ pro-

Ineffective Proofs

2003

Polynomial-time Algorithms from Ineffective Proofs

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Abstract

We present a constructive procedure for extracting polynomial-time realizers from ineffective proofs of Π^0_2*-theorems in feasible analysis. By ineffective proof we mean a proof which involves the non-computational principle weak König's lemma* WKL*, and by feasible analysis we mean Cook and Urquhart's system* CPV^ω *plus quantifier-*

same property of IS^1_2 that the provably recursive functions are polynomial-time computable. Cook and Urquhart then developed variants of Kreisel's modified realizability and Gödel's functional interpretation for the system IPV^ω. The latter via negative translation applies also to CPV^ω. Given a proof of a Π^0_2-theorem of IPV^ω or CPV^ω, these interpretations provide a simple procedure for extracting from this proof a polynomial-time algorithm realizing the theorem.

$$\mathcal{B}(Y, W, z) = \begin{cases} z & \text{if } |Y\hat{w}_z| \leq |w_z| \\ & \text{or } |w_z| \neq |z| \\ \mathcal{B}(Y, W, z1) & \text{otherwise,} \end{cases} \tag{2}$$

Theorem 5.1 *The theory* $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} + \Pi^0_1\text{-}\mathsf{WKL}^\omega$ *has a functional interpretation (via negative translation) in* $\mathsf{IPV}^\omega + \mathsf{BND} + (2)$.

Lemma 4.4 *Let* $t[x, \alpha]$ *be a term of* $\mathcal{L}(\mathsf{IPV}^\omega) \cup \{\mathcal{B}\}$ *of type* $\mathbb{N}$, *having as only free-variables* x *and* α, *such that (for simplicity)* $\mathcal{B}$ *is always applied to zero on the third argument. Then, there exists a polynomial-time computable function* h *(with 0-1 oracle) such that for all input* x *and for all 0-1 oracles* α, $h(x, \alpha) = t[x, \alpha]$.

2012

On the Relation Between Various Negative Translations

Gilda Ferreira* and Paulo Oliva†

Several proof translations of classical mathematics into intuitionistic (or even minimal) mathematics have been proposed in the literature over the past century. These are normally referred to as *negative translations* or *double-negation translations*. Amongst those, the most commonly cited are translations due to Kolmogorov, Gödel, Gentzen, Kuroda and Krivine (in chronological order). In this paper we propose a framework for explaining how these different translations are related to each other. More precisely, we define a notion of a (modular) simplification starting from Kolmogorov translation, which leads to a partial order between different negative translations. In this derived ordering, Kuroda, Krivine and Gödel-Gentzen are minimal elements. A new minimal translation is introduced.

Bar Recursion

2003

MODIFIED BAR RECURSION AND CLASSICAL DEPENDENT CHOICE

ULRICH BERGER AND PAULO OLIVA

Abstract. We introduce a variant of Spector's bar recursion in finite types (which we call "modified bar recursion") to give a realizability interpretation of the classical axiom of dependent choice allowing for the extraction of witnesses from proofs of ∀∃-formulas in classical analysis. As another application, we show that the fan functional can be defined by modified bar recursion together with a version of bar recursion due to Kohlenbach. We also show that the type structure $\mathcal{M}$ of strongly majorizable functionals is a model for modified bar recursion.

2010

Selection Functions, Bar Recursion, and Backward Induction

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Bar recursion arises in constructive mathematics, logic, proof theory and higher-type computability theory. We explain bar recursion in terms of sequential games, and show how it can be naturally understood as a generalisation of the principle of backward induction that arises in game theory. In summary, bar recursion calculates optimal plays

2015

BAR RECURSION AND PRODUCTS OF SELECTION FUNCTIONS

MARTÍN ESCARDÓ AND PAULO OLIVA

Abstract. We show how two iterated products of selection functions can both be used in conjunction with system T to interpret, via the dialectica interpretation and modified realizability, full classical analysis. We also show that one iterated product is equivalent over system T to Spector's bar recursion, whereas the other is T-equivalent to modified bar recursion. Modified bar recursion itself is shown to arise directly from the iteration of a different binary product of 'skewed' selection functions. Iterations of the dependent binary products are also considered but in all cases are shown to be T-equivalent to the iteration of the simple products.

Functional Interpretations

Definition 3.1 (Functional Interpretation). *The interpretation associates to each formula $A \in \mathcal{L}(\mathbf{HA}^\omega)$ (by induction on the logical structure of A) another formula $(A)^D$ of the form $\exists x \forall y A_D(x,y)$, where A_D is quantifier free, in the following manner:*

$$A^D :\equiv A, \text{ for atomic formulas } A,$$

and assuming $A^D = \exists x \forall y A_D(x,y)$ and $B^D = \exists z \forall w B_D(z,w)$ we define

$$\begin{aligned}
(A \wedge B)^D &:\equiv \exists x, z \forall y, w (A_D(x,y) \wedge B_D(z,w)), \\
(A \vee B)^D &:\equiv \exists p^0 \exists x, z \forall y, w((p = 0 \to A_D(x,y)) \wedge (p \neq 0 \to B_D(z,w))), \\
(A \to B)^D &:\equiv \exists \Psi, \Phi \forall x, w(A_D(x, \Phi x w) \to B_D(\Psi x, w)), \\
(\exists z A(z))^D &:\equiv \exists z, x \forall y A_D(x,y,z), \\
(\forall z A(z))^D &:\equiv \exists \Psi \forall z, y A_D(\Psi z, y, z),
\end{aligned}$$

where the types of Ψ and Φ can be inferred. We define $\neg A$ as $A \to 0 = 1$.

Using the relation $\geq^*$, the *monotone functional interpretation (m.f.i.)* of a formula A (having functional interpretation $\exists x^\rho \forall y^\tau A_D(x,y)$) is defined as

$$\exists x^* \exists x \leq^*_\rho x^* \forall y A_D(x,y).$$

2004

Bounded Functional Interpretation

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Abstract

We present a new functional interpretation, based on a novel assignment of formulas. In contrast with Gödel's functional "Dialectica" interpretation, the new interpretation does not care for precise witnesses of existential statements, but only for bounds for them. New principles are supported by our interpretation, including (a version of) the FAN theorem, weak König's lemma and the lesser limited principle of omniscience. Conspicuous among these principles are also refutations of some laws of classical logic. Notwithstanding, we end up discussing some applications of the new interpretation to theories of classical arithmetic and analysis.

2006

Unifying Functional Interpretations

Paulo Oliva

Abstract This article presents a parametrized functional interpretation. Depending on the choice of two parameters one obtains well-known functional interpretations such as Gödel's Dialectica interpretation, Diller-Nahm's variant of the Dialectica interpretation, Kohlenbach's monotone interpretations, Kreisel's modified realizability, and Stein's family of functional interpretations. A functional interpretation consists of a formula interpretation and a soundness proof. I show that all these interpretations differ only on two design choices: first, on the number of counterexamples for A which became witnesses for $\neg A$ when defining the formula interpretation and, second, the inductive information about the witnesses of A which is considered in the proof of soundness. Sufficient conditions on the parameters are also given which ensure the soundness of the resulting functional interpretation. The relation between the parametrized interpretation and the recent bounded functional interpretation is also discussed.

2006

Unifying Functional Interpretations

Paulo Oliva

$$
\begin{aligned}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \vee B|^{\boldsymbol{x},\boldsymbol{v},n}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vee_n |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \to B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} &:\equiv \forall \boldsymbol{y} \sqsubset \boldsymbol{g}\boldsymbol{x}\boldsymbol{w}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{f}\boldsymbol{x}}_{\boldsymbol{w}} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} &:\equiv |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{y}} &:\equiv |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{aligned}
$$

… and more …

Thank you, Ulrich,
and happy 60th birthday!