The Selection Monad Transformer and Higher-Order Games

Winter Meeting in Lisbon - Mathematics & Applications
18/12/2023

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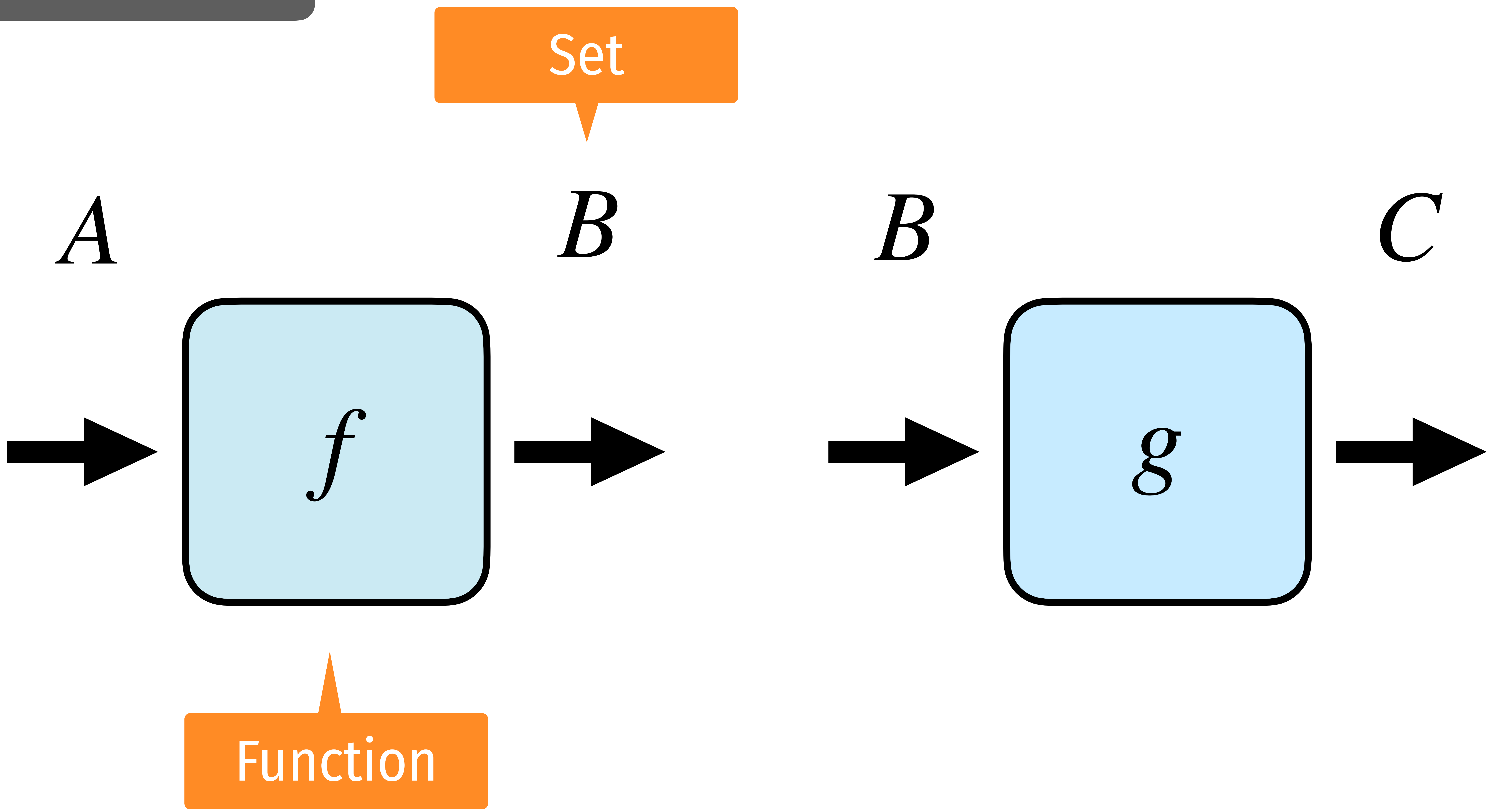
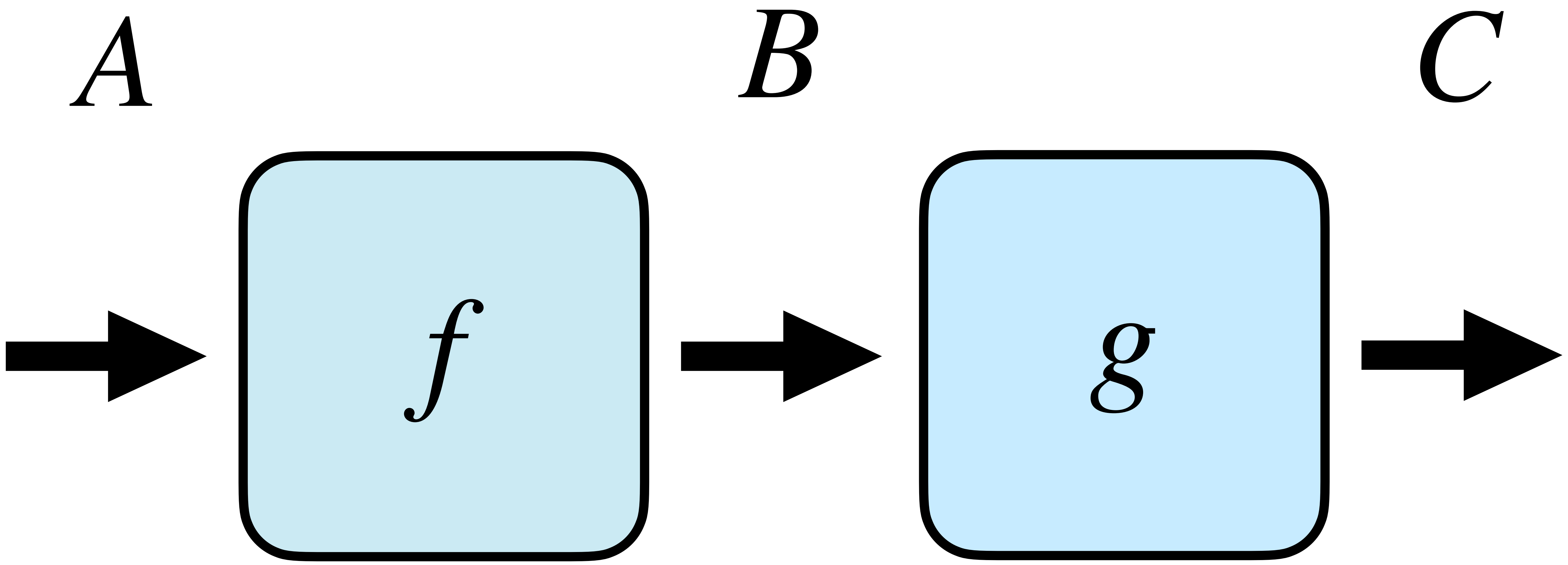
Modularity and Composition

Composition



Composition



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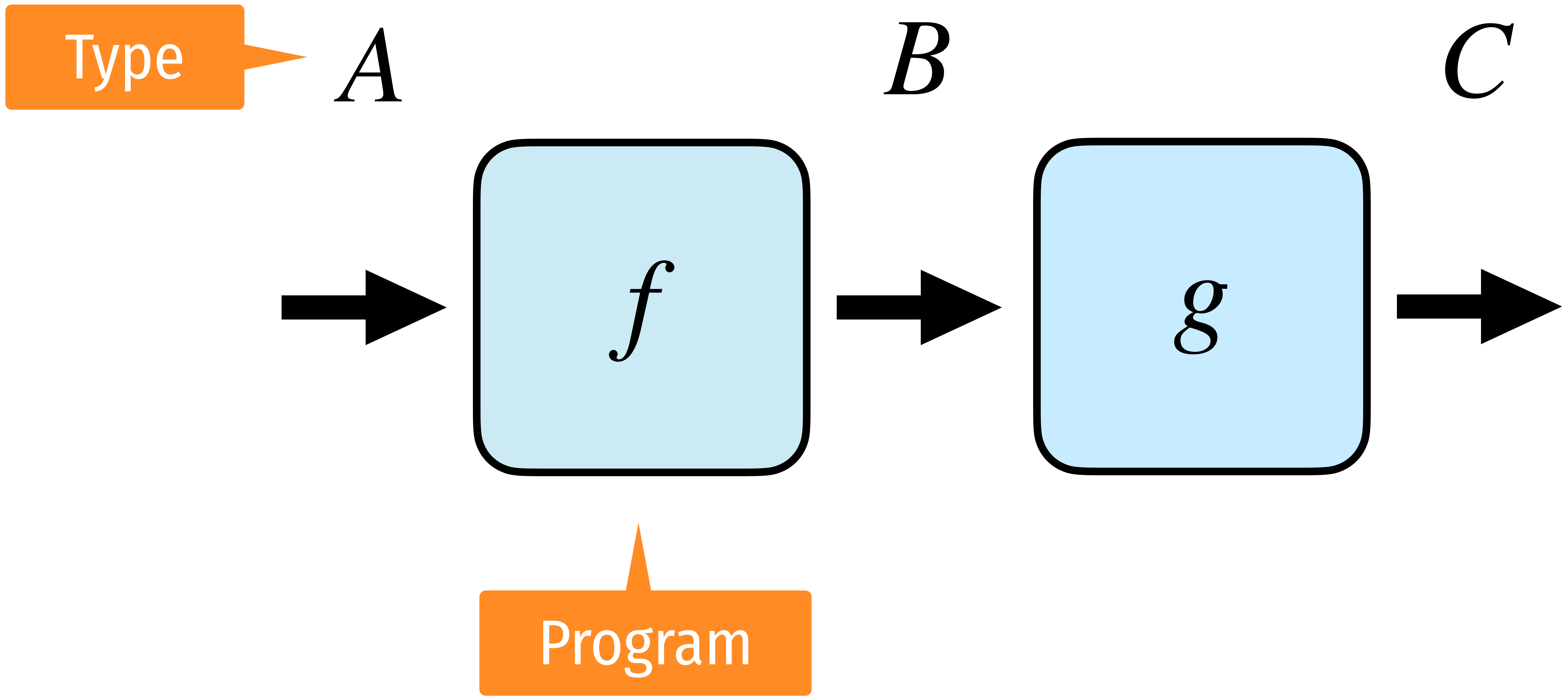
Composition of Programs



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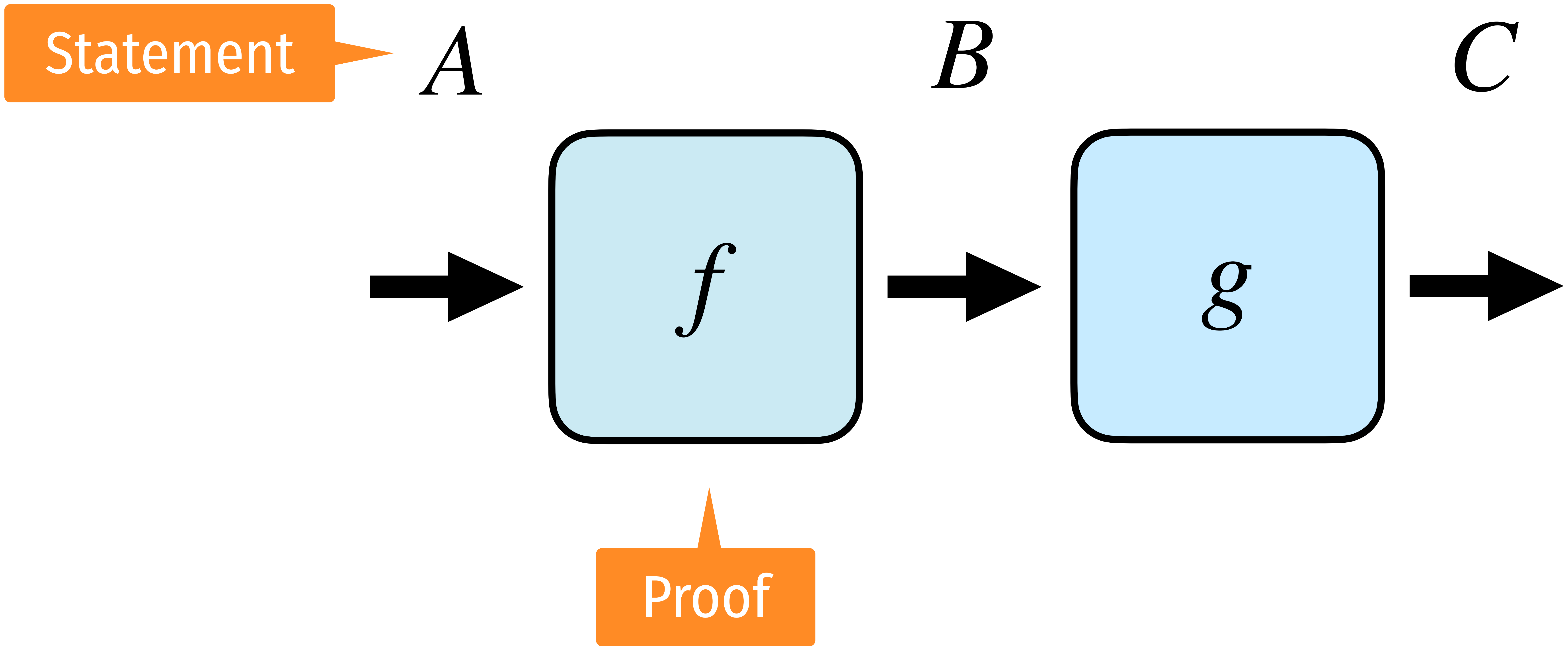
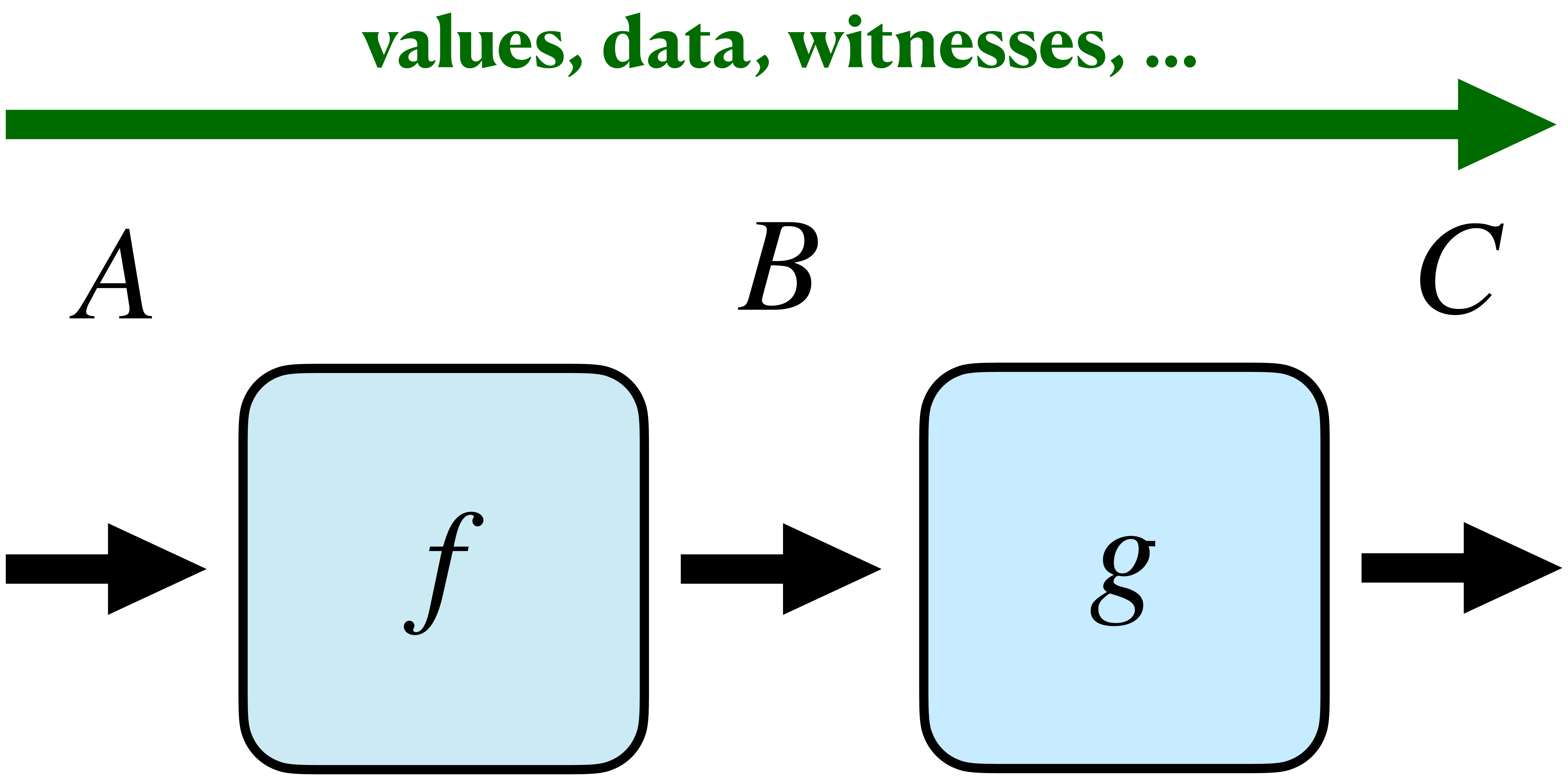
Composition of Proofs

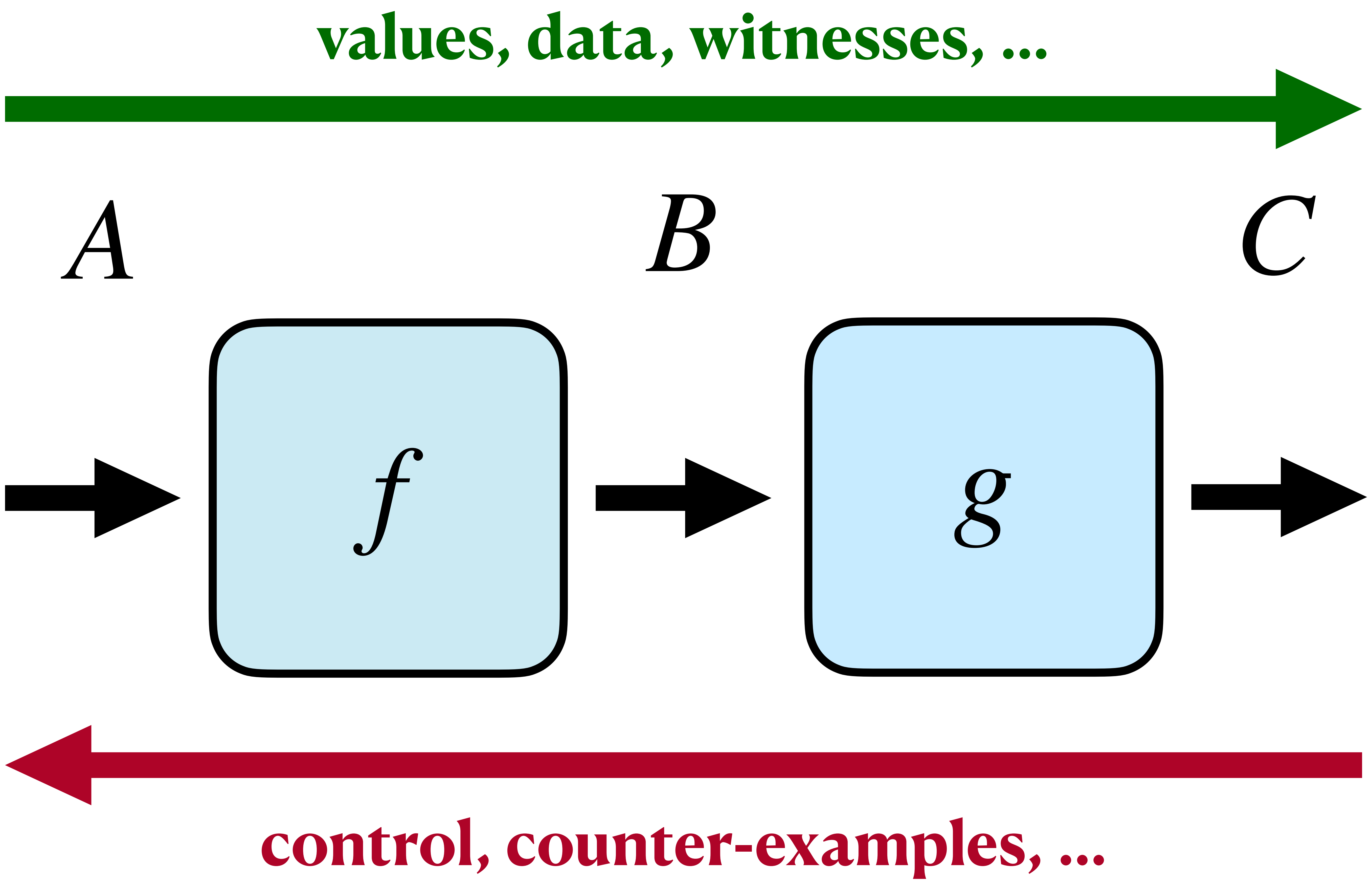


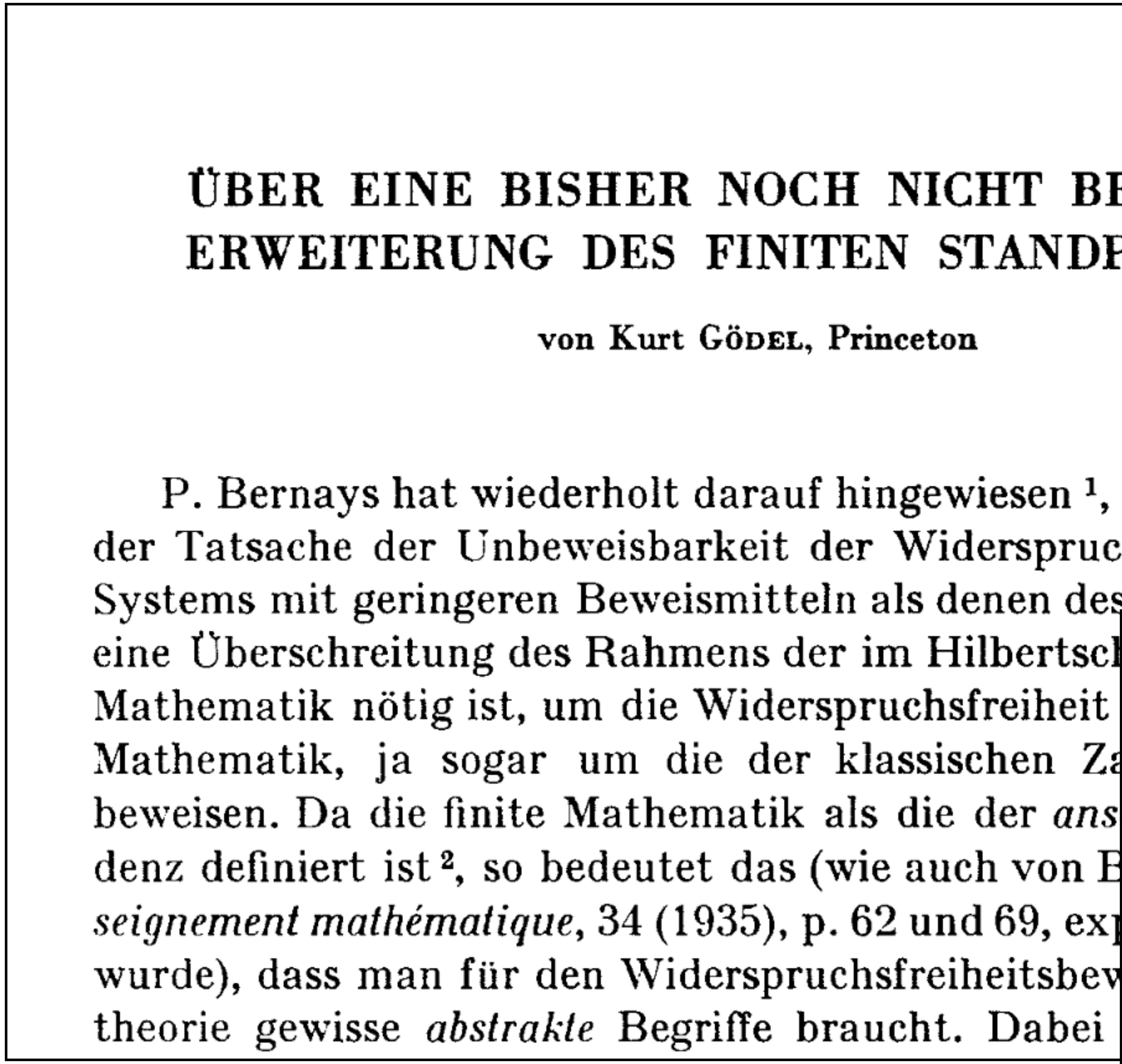
Flow and Counter-Flow

Flow



Counter-Flow





ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

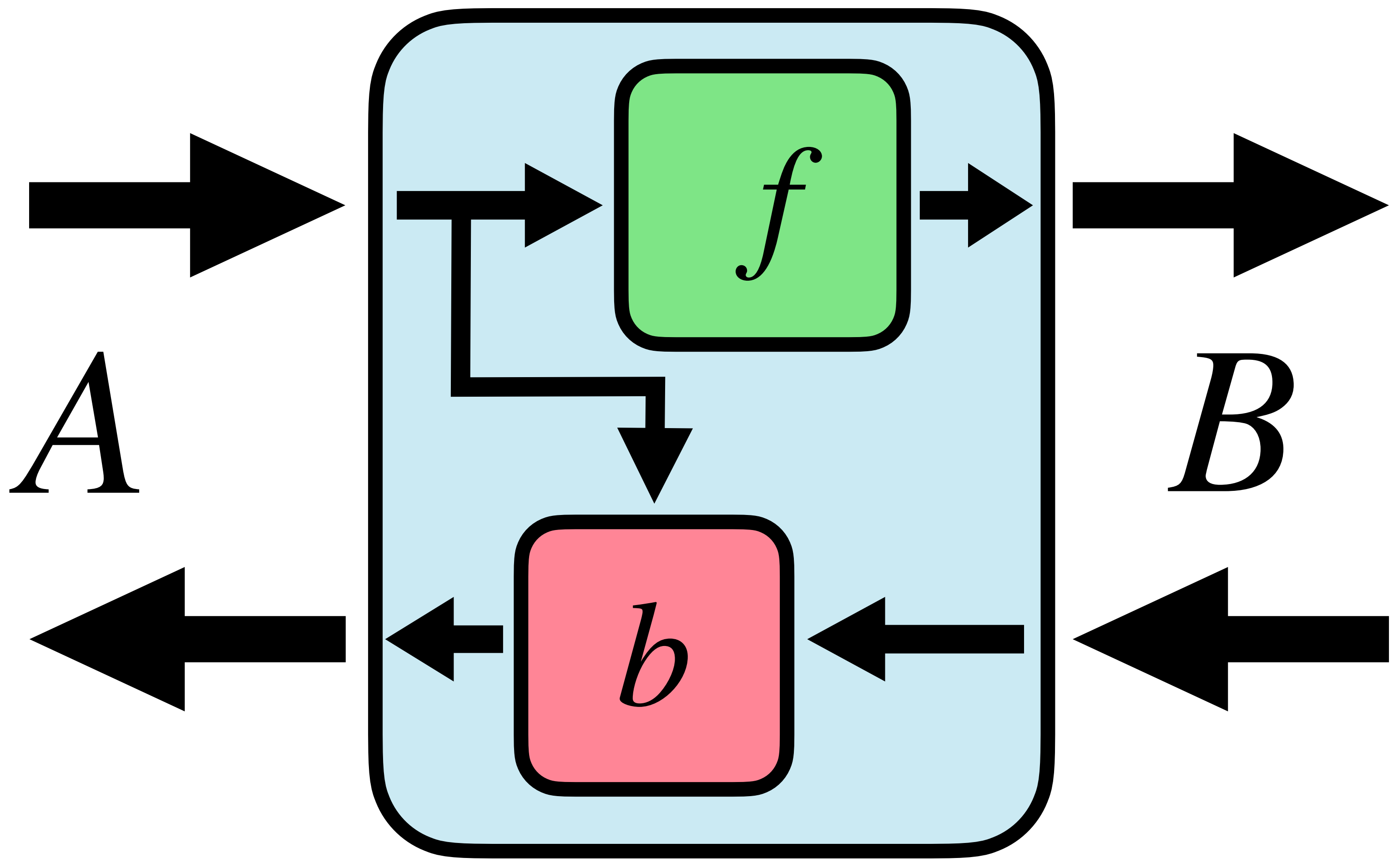
von Kurt Gödel, Princeton

P. Bernays hat wiederholt darauf hingewiesen [1], dass angesichts der Tatsache der Unbeweisbarkeit der Widerspruchsfreiheit eines Systems mit geringeren Beweismitteln als denen des Systems selbst eine Überschreitung des Rahmens der im Hilbertsc
Mathematik nötig ist, um die Widerspruchsfreiheit
Mathematik, ja sogar um die der klassischen Za
beweisen. Da die finite Mathematik als die der *ans*
denz definiert ist [2], so bedeutet das (wie auch von B
seignement mathématique, 34 (1935), p. 62 und 69, exp
wurde), dass man für den Widerspruchsfreiheitsbew
theorie gewisse *abstrakte* Begriffe braucht. Dabei

Volume 12, Issue 3-4
December 1958
Pages 280-287

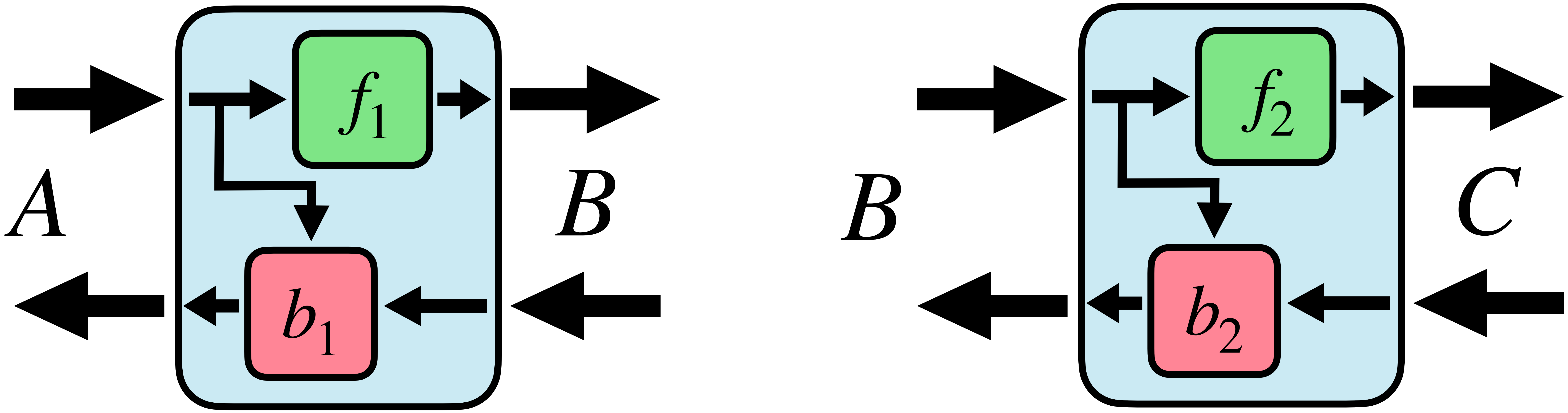
Flow and Counter-Flow

Relation $A \subseteq A^+ \times A^-$

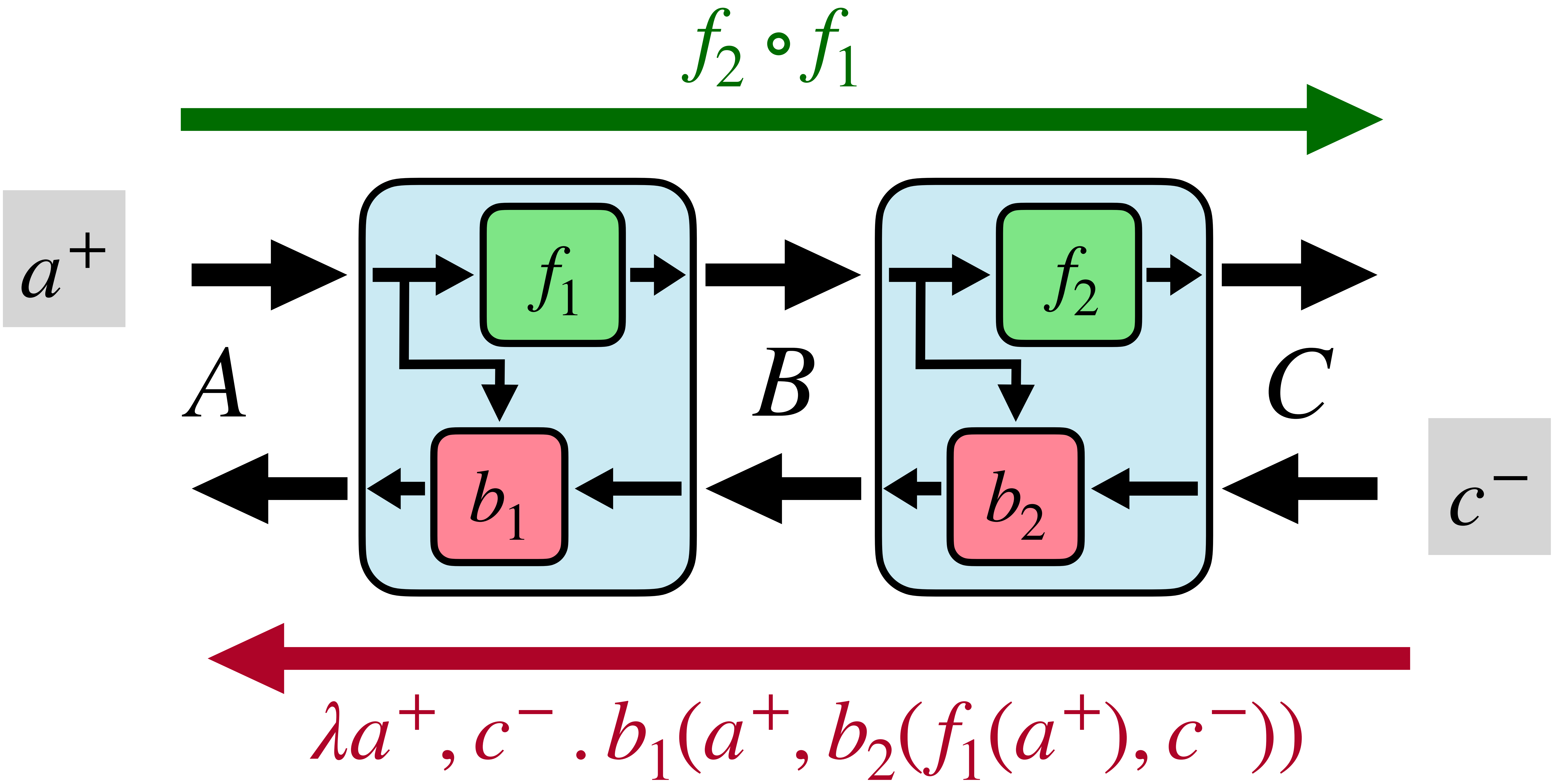


Pair of functions $(f, b) : (A^+ \to B^+) \times (A^+ \times B^- \to A^-)$

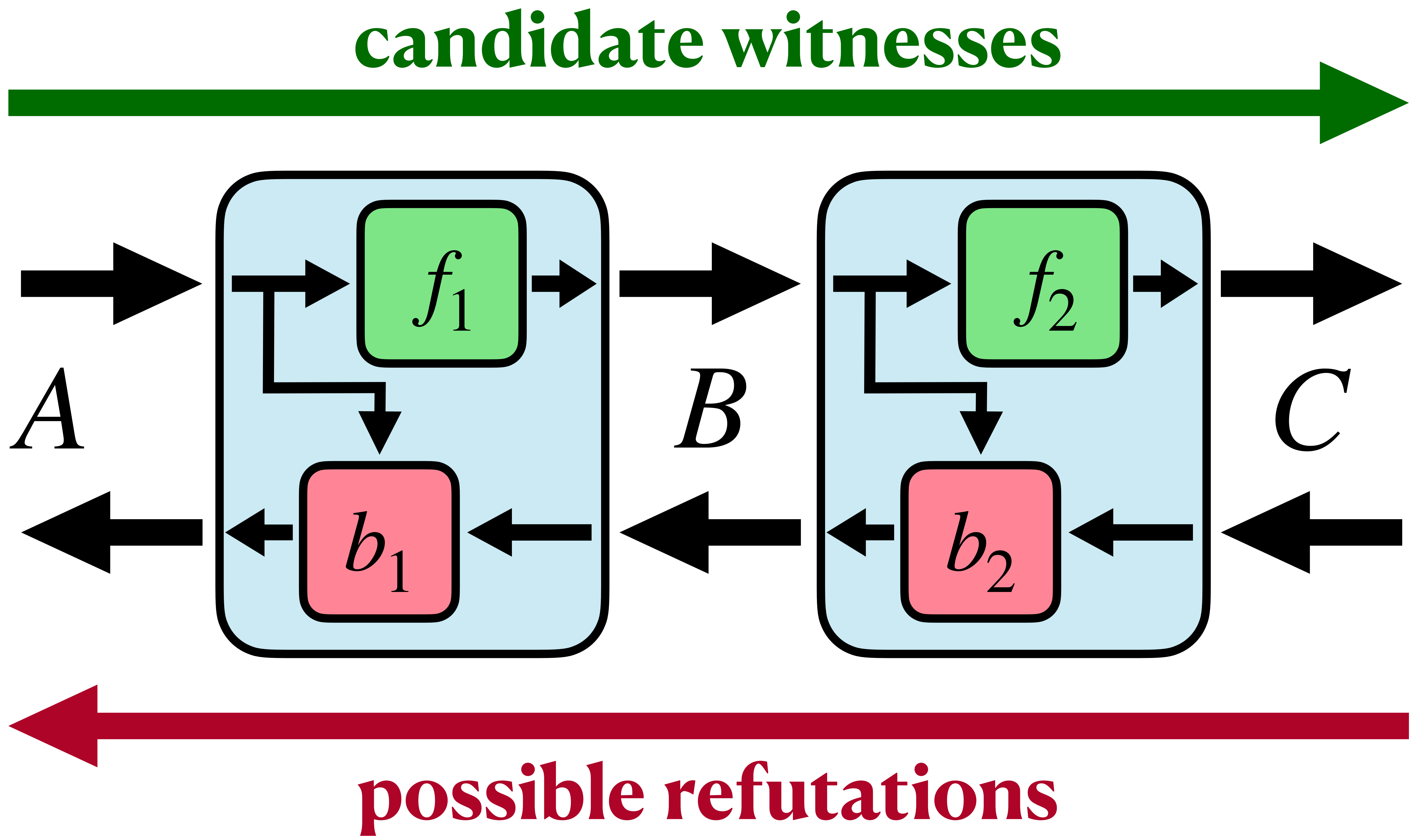
Composition



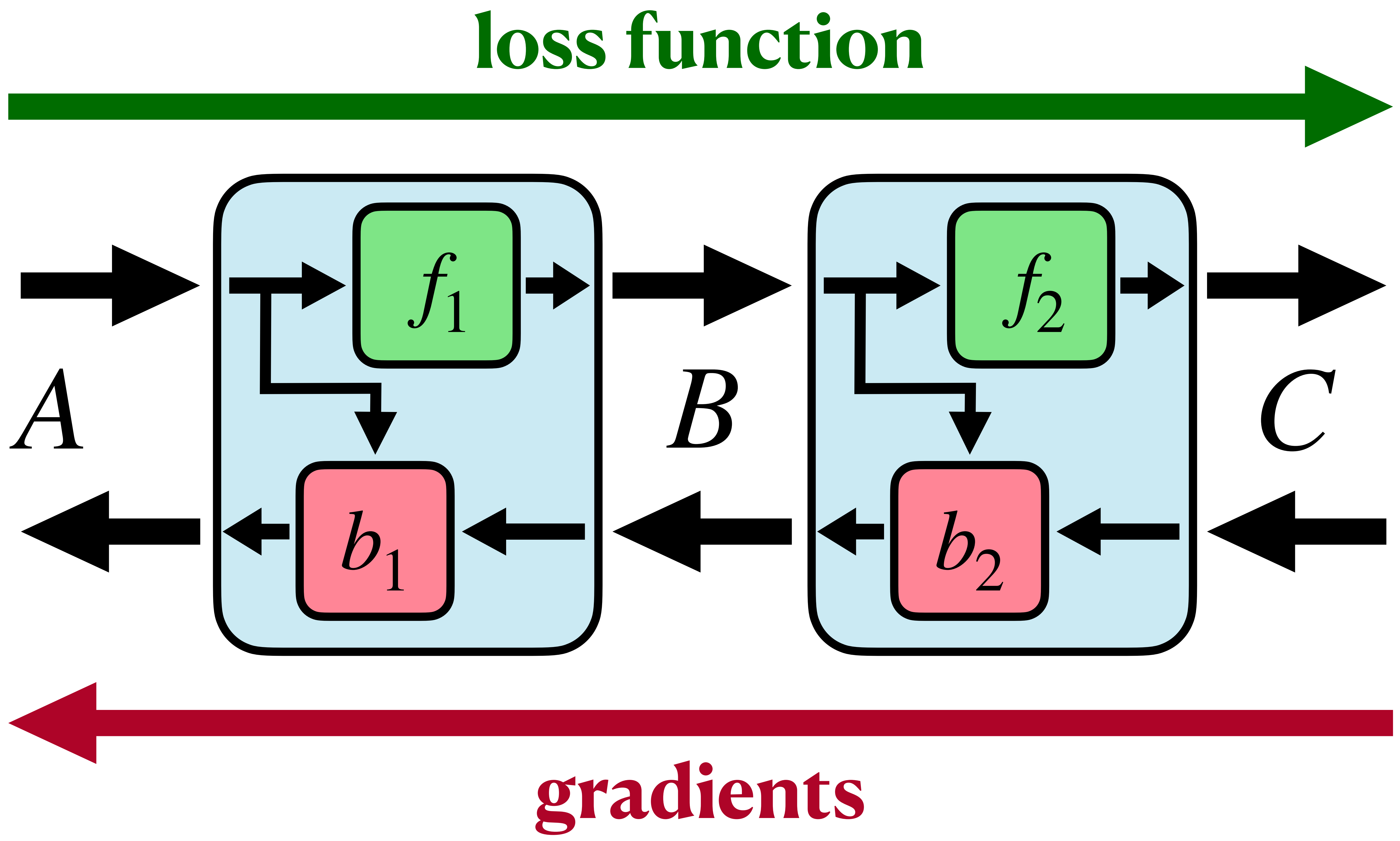
Composition



Gödel's Dialectica Interpretation



Neural Networks and Back Propagation

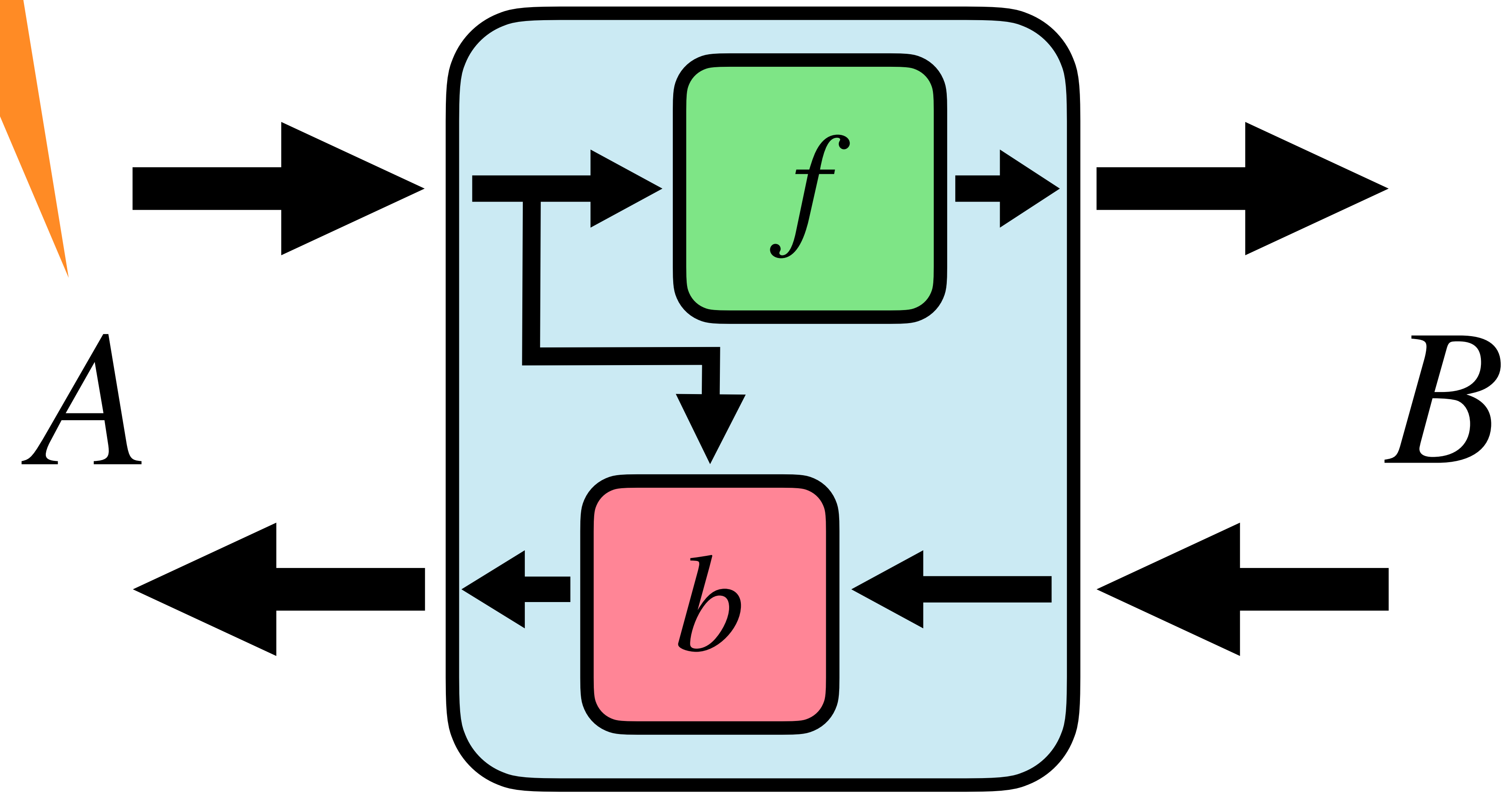


Games

Games

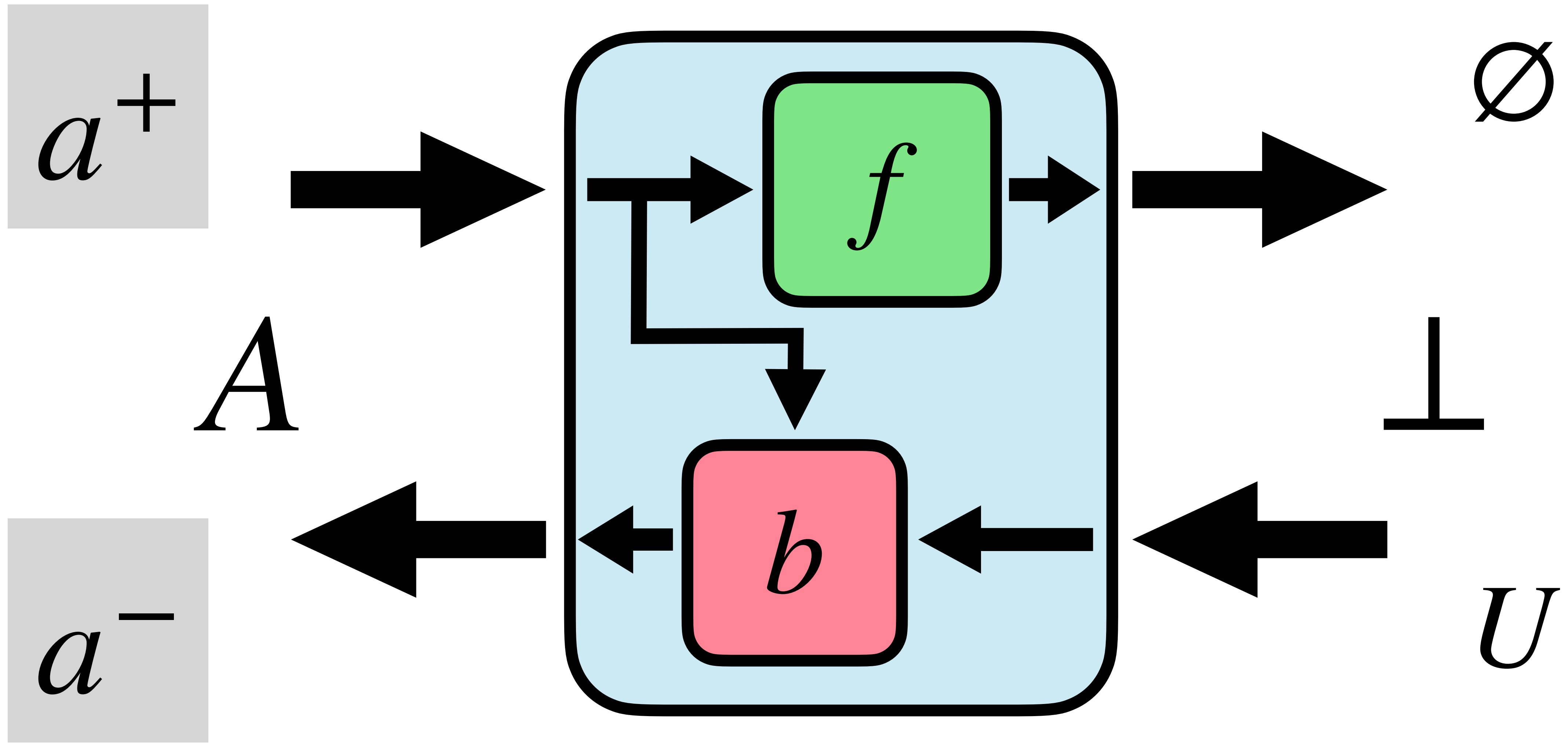
Play game B relative to game A

Relation $A \subseteq A^+ \times A^-$



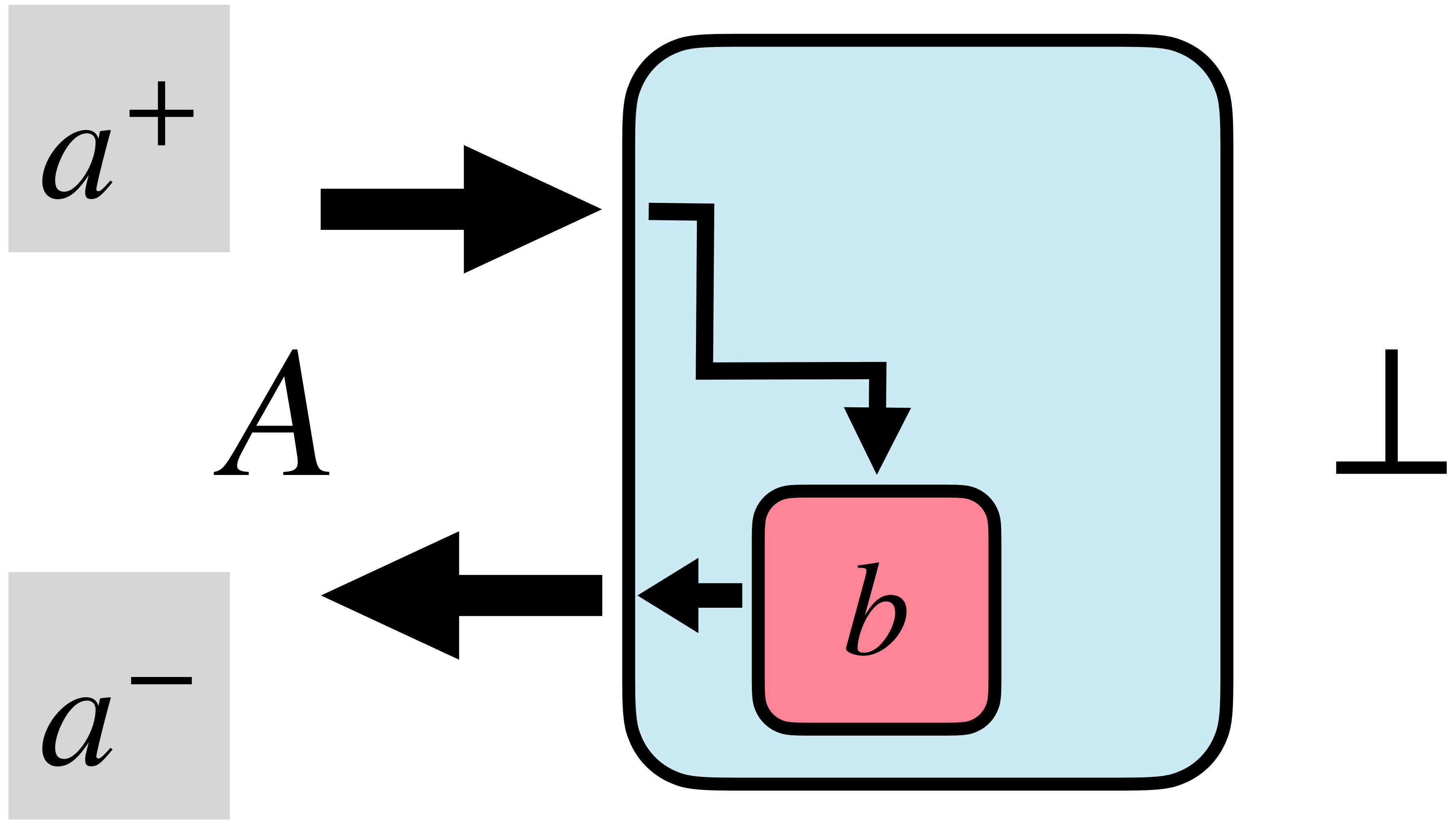
Game
$A^+ =$ Eloise
$A^- =$ Abelard

The $\neg A$ Game



The $\neg A$ Game

Game $\neg A \subseteq (A^+ \to A^-) \times A^+$



The $\neg\neg A$ Game

$$\text{Game } \neg\neg A \subseteq \underbrace{((A^+ \to A^-) \to A^+)}_{(X \to R) \to X} \times (A^+ \to A^-)$$

selection functions

Selection Functions and Quantifiers

$$\varepsilon : (X \to R) \to X$$

selection function

$$\varphi : (X \to R) \to R$$

quantifier

$$\varphi(p) = p(\varepsilon(p))$$

attainability

Selection Functions and Quantifiers

$$\varepsilon : (X \to \mathbb{R}) \to X$$

selection function

$$\exists : (X \to \mathbb{B}) \to \mathbb{B}$$

quantifier

$$\exists(p) = p(\varepsilon(p))$$

attainability

Selection Functions and Quantifiers

$$\mathrm{argmax} : (X \to \mathbb{R}) \to X$$

selection function

$$\max : (X \to \mathbb{R}) \to \mathbb{R}$$

quantifier

$$\max(p) = p(\mathrm{argmax}(p))$$

attainability

Selection and Continuation Monad

$$J_R X = (X \to R) \to X$$

selection monad

$$K_R X = (X \to R) \to R$$

continuation monad

$\eta^K_X \colon X \to K_R X$

$\eta^J_X \colon X \to J_R X$

$\beta^K_X \colon K_R X \to (X \to K_R Y) \to K_R Y$

$\beta^J_X \colon J_R X \to (X \to J_R Y) \to J_R Y$

Tensor of Quantifiers and Selection Functions

We can compose selection functions and quantifiers!

$$\bigotimes : K_R X \times K_R Y \to K_R(X \times Y)$$

$$\bigotimes : J_R X \times J_R Y \to J_R(X \times Y)$$

Higher-Order Games

Higher-Order Game

R outcomes

X_i possible moves at round i

$q : \Pi_i X_i \to R$ outcome function (outcome given a play)

$\varphi_i : (X_i \to R) \to R$ quantifiers (preferred outcome given context)

PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2011) **467**, 1519–1545
doi:10.1098/rspa.2010.0471
Published online 1 December 2010

REVIEW

Sequential games and optimal strategies

BY MARTÍN ESCARDÓ[1] AND PAULO OLIVA[2,*]

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University of London, London E1

This article gives an overview of recent work on the th
explain the intuition behind these higher type objects,
sequential game whose optimal strategies can be com
selection functions. Several instances of this game are con
as fixed point theory, topology, game theory, higher type
These examples are intended to illustrate how the fund
strategies based on products of selection functions perm

Keywords: selection function; Nash equilibrium; backw
bar recursion; Tychonoff theo

julesh
@_julesh_

The paper Sequential Games and Optimal Strategies by Martín Escardó and Paulo Oliva, which single-handedly got me into game theory and convinced me to apply to do a PhD with Paulo
royalsocietypublishing.org/doi/full/10.10...

> **Paul Graham** @paulg · Sep 14, 2019
> If you had to pick one thing you've read that changed the course of your life, what would it be? (ht @rivatez for the question.)

9:04 PM · Sep 14, 2019

Higher-Order Game

R — outcomes

X_i — moves

$q : \Pi_i X_i \to R$ — outcome function

$\varphi_i : (X_i \to R) \to R$ — goal functions

$\left(\bigotimes \varphi_i\right)(q)$ — optimal outcome

$\left(\bigotimes \varepsilon_i\right)(q)$ — optimal strategy

Monadic Strategies

Selection Monad Transformer

$$J_R X = (X \to R) \to X$$

selection monad

Given any monad TX such that R is a T-algebra ($\alpha \colon TR \to R$)...

$$J_R^T X = (X \to R) \to TX$$

... also a monad

Recent & Current Work

- Recent work:
  - Use of **powerset monad** for interpretation of countable choice via Herbrand interpretation
  - Extension of the framework to deal with **dependent types**
- Current work:
  - **Probability monad**: Sub-optimal or irrational players
  - **Reader monad**: Alpha-beta pruning

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