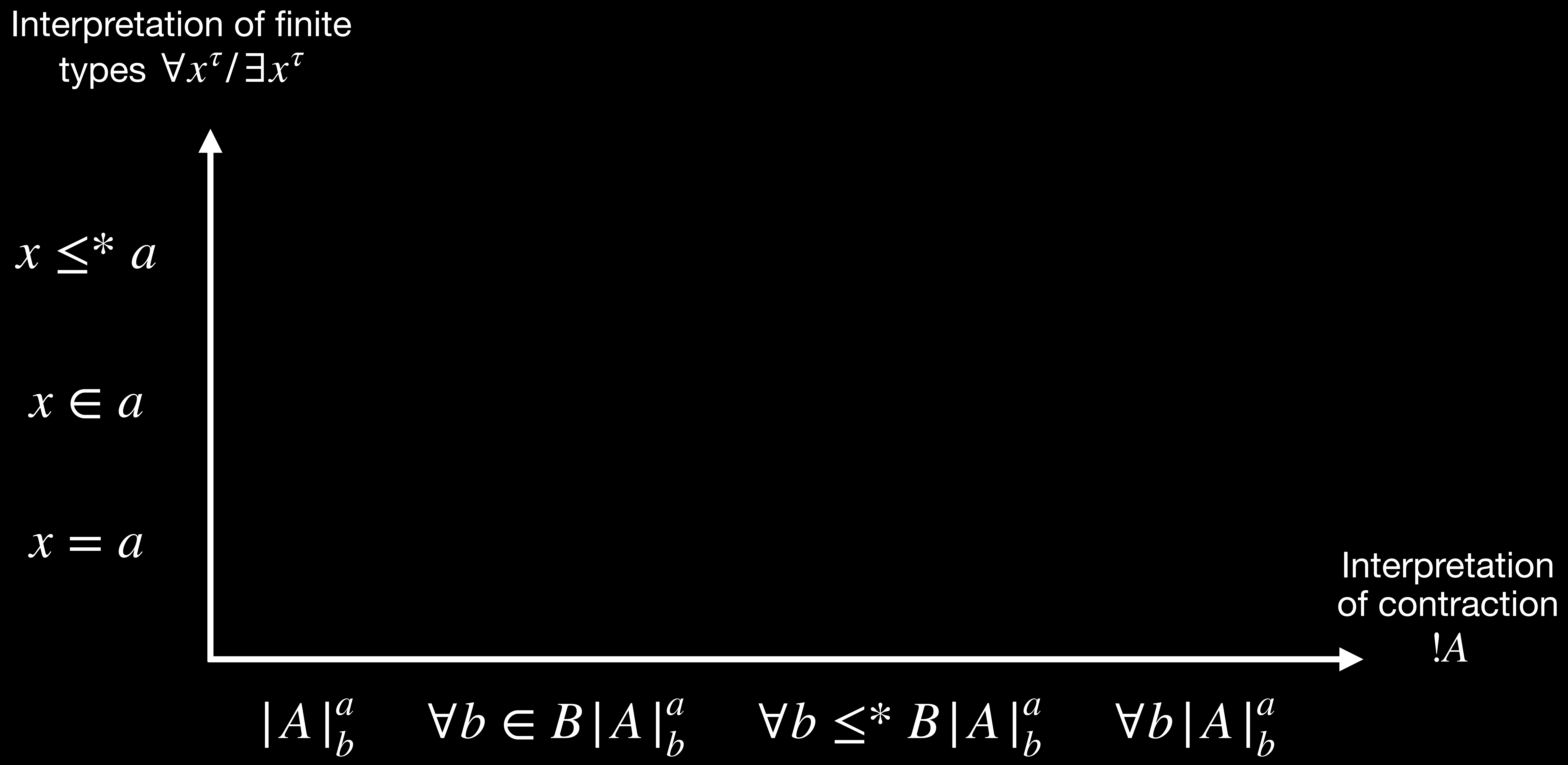
Bounding Interpretations with Abstract Spaces

Joint work with F. Ferreira
Building on joint work with B. Dinis

Workshop on Proof Mining, Darmstadt
Paulo Oliva — 04/09/2024



Interpretation of finite types $\forall x^{\tau} / \exists x^{\tau}$
$x \leq^* a$
$x \in a$
$x = a$
Interpretation of contraction $!A$
$|A|_b^a$
$\forall b \in B \, |A|_b^a$
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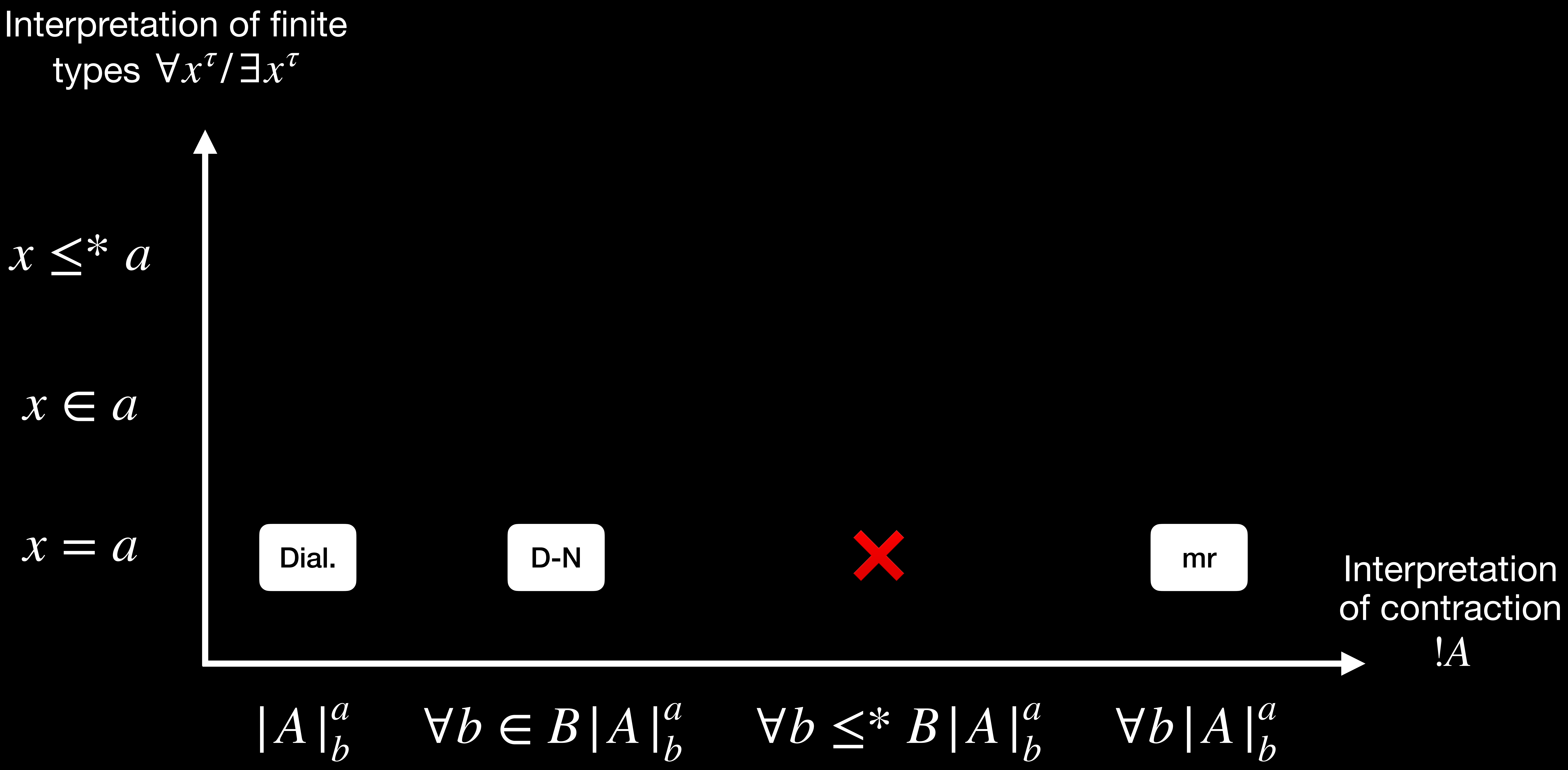
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Dial.
D-N
mr
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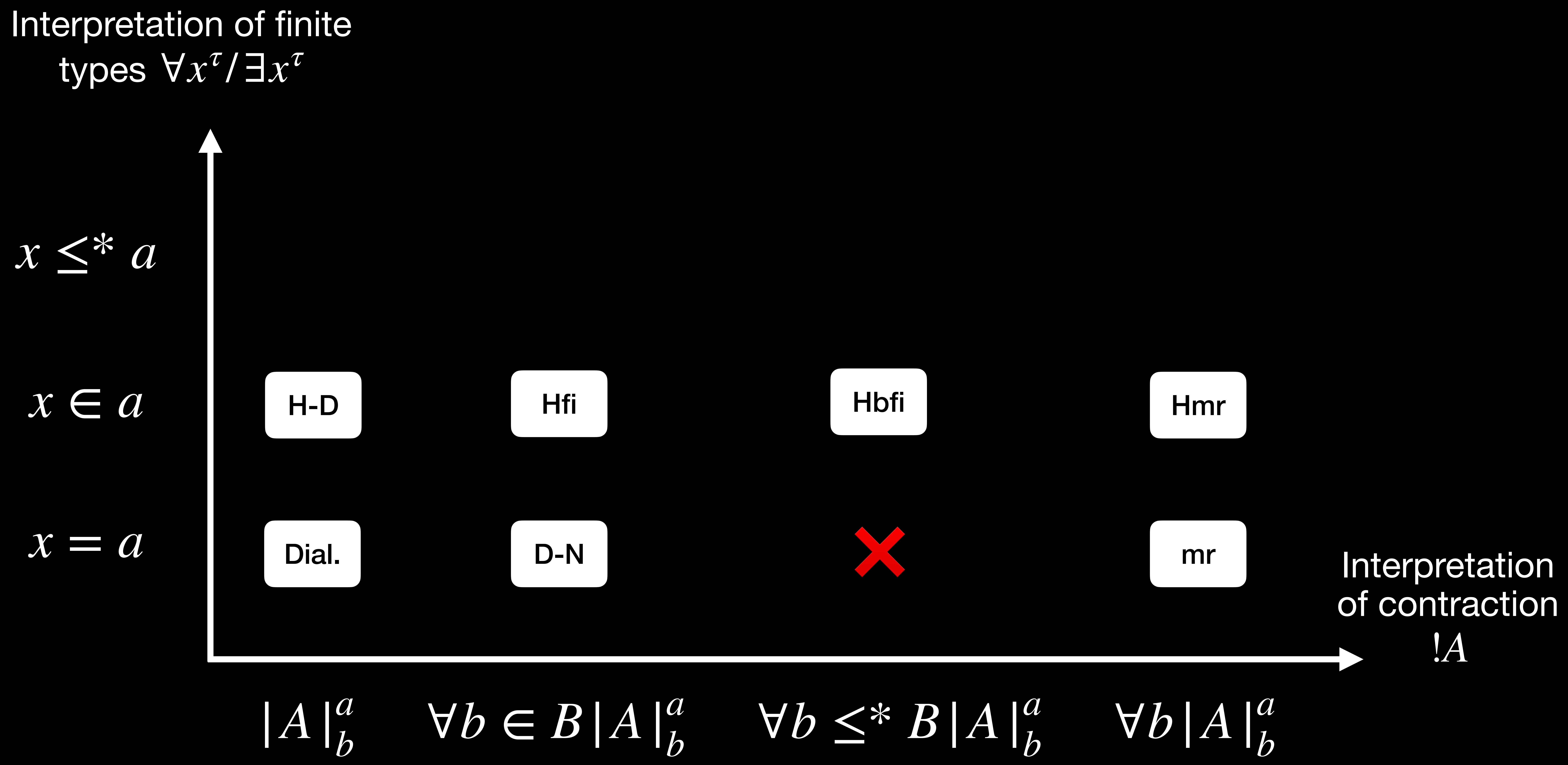
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$x \leq^{*} a$
$x \in a$
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H-D
Hfi
Hbfi
Hmr
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$|A|^{a}_{b}$
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Interpretation of finite types $\forall x^\tau / \exists x^\tau$

Interpretation of contraction $!A$

	$\lvert A \rvert_b^a$	$\forall b \in B \lvert A \rvert_b^a$	$\forall b \leq^* B \lvert A \rvert_b^a$	$\forall b \lvert A \rvert_b^a$
$x \leq^* a$	❌	💡	bfi	bmr
$x \in a$	H-D	Hfi	Hbfi	Hmr
$x = a$	Dial.	D-N	❌	mr

Interpretation of finite types $\forall x^\tau / \exists x^\tau$

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$x = a$	Dial.	D-N	❌	mr

Interpretation of contraction $!A$

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A parametrised functional interpretation of Heyting arithmetic

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A R T I C L E I N F O

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A B S T R A C T

This paper presents an *abstract* parametrised functional interpretation of $\mathbf{WE\text{-}HA}^{\omega}$. It is based on families of parameters allowing for different degrees of freedom on the design of *concrete* interpretations. In this way, we are able to generalise previous work on unifying functional interpretations by including in the unification the more recent bounded and Herbrandized functional interpretations.

$\mathrm{wt}(\tau)$	$x \prec^\tau a$	$\mathrm{m}^{\mathbb{B}}_\tau(b,x,y)$ / $\mathrm{m}^{\mathbb{N}}_\tau(f,a)$	$\mathrm{bt}(\boldsymbol{\tau})$	$\forall \boldsymbol{x} \sqsubset_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	$\mathrm{W}_\tau(x)$	**Interpretation**
τ	$x =_\tau a$	$\mathrm{if}_\tau(b,x,y)$ / $f(a)$	$\boldsymbol{\tau}$	$A[\boldsymbol{a}/\boldsymbol{x}]$	true	Dialectica
τ	$x =_\tau a$	$\mathrm{if}_\tau(b,x,y)$ / $f(a)$	ε	$\forall \boldsymbol{x}^{\boldsymbol{\tau}} A$	true	Modified realizability
τ	$x =_\tau a$	$\mathrm{if}_\tau(b,x,y)$ / $f(a)$	$\boldsymbol{\tau}$	$\forall \boldsymbol{x} \leq^*_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	true / $x \leq^*_\tau x$	(*combination not sound*)
τ	$x =_\tau a$	$\mathrm{if}_\tau(b,x,y)$ / $f(a)$	$\boldsymbol{\tau}^*$	$\forall \boldsymbol{x} \in_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	true	Diller-Nahm
τ	$x \leq^*_\tau a$	$\max_\tau(x,y)$ / $f(a)$	$\boldsymbol{\tau}$	$A[\boldsymbol{a}/\boldsymbol{x}]$	$x \leq^*_\tau x$	(*combination not sound*)
τ	$x \leq^*_\tau a$	$\max_\tau(x,y)$ / $f(a)$	ε	$\tilde{\forall} \boldsymbol{x}^{\boldsymbol{\tau}} A$	$x \leq^*_\tau x$	Bounded modified realizability
τ	$x \leq^*_\tau a$	$\max_\tau(x,y)$ / $f(a)$	$\boldsymbol{\tau}$	$\tilde{\forall} \boldsymbol{x} \leq^*_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	$x \leq^*_\tau x$	Bounded functional interpretation
τ	$x \leq^*_\tau a$	$\max_\tau(x,y)$ / $f(a)$	$\boldsymbol{\tau}^*$	$\tilde{\forall} \boldsymbol{x} \in_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	$x \leq^* x$	**Bounded Diller-Nahm**
τ^*	$x \in_\tau a$	$x \cup y$ / $\bigcup_{z \in a} fz$	$\boldsymbol{\tau}$	$A[\boldsymbol{a}/\boldsymbol{x}]$	true	Herbrand Dialectica ($\simeq$ Dialectica)
τ^*	$x \in_\tau a$	$x \cup y$ / $\bigcup_{z \in a} fz$	ε	$\forall \boldsymbol{x}^{\boldsymbol{\tau}} A$	true	Herbrand realizability (for IL)
τ^*	$x \in_\tau a$	$x \cup y$ / $\bigcup_{z \in a} fz$	$\boldsymbol{\tau}$	$\tilde{\forall} \boldsymbol{x} \leq^*_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	$x \leq^*_\tau x$	**Herbrandized bfi**
τ^*	$x \in_\tau a$	$x \cup y$ / $\bigcup_{z \in a} fz$	$\boldsymbol{\tau}^*$	$\forall \boldsymbol{x} \in_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	true	Herbrand Diller-Nahm

Fig. 1. Summary of instantiations (with the two novel interpretations in **bold**).

$\mathrm{wt}(\tau)$	$x \prec^\tau a$	$\mathrm{m}^{\mathbb{B}}_\tau(b,x,y)$ / $\mathrm{m}^{\mathbb{N}}_\tau(f,a)$	$\mathrm{bt}(\boldsymbol{\tau})$	$\forall \boldsymbol{x} \sqsubset_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	$\mathrm{W}_\tau(x)$	**Interpretation**
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τ	$x =_\tau a$	$\mathrm{if}_\tau(b,x,y)$ / $f(a)$	$\boldsymbol{\tau}^*$	$\forall \boldsymbol{x} \in_{\boldsymbol{\tau}} \boldsymbol{a}\, A$	true	Diller-Nahm
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SOME LOGICAL METATHEOREMS WITH APPLICATIONS IN FUNCTIONAL ANALYSIS

ULRICH KOHLENBACH

ABSTRACT. In previous papers we have developed proof-theoretic techniques for extracting effective uniform bounds from large classes of ineffective existence proofs in functional analysis. Here 'uniform' means independence from parameters in compact spaces. A recent case study in fixed point theory systematically yielded uniformity even w.r.t. parameters in metrically bounded (but noncompact) subsets which had been known before only in special cases. In the present paper we prove general logical metatheorems which cover these applications to fixed point theory as special cases but are not restricted to this area at all. Our theorems guarantee under general logical conditions such strong uniform versions of non-uniform existence statements. Moreover, they provide algorithms for actually extracting effective uniform bounds and transforming the original proof into one for the stronger uniformity result. Our metatheorems deal with general classes of spaces like metric spaces, hyperbolic spaces, CAT(0)-spaces, normed linear spaces, uniformly convex spaces, as well as inner product spaces.

The abstract type of the real numbers

Fernando Ferreira

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Abstract

In finite type arithmetic, the real numbers are represented by rapidly converging Cauchy sequences of rational numbers. Ulrich Kohlenbach introduced abstract types for certain structures such as metric spaces, normed spaces, Hilbert spaces, etc. With these types, the elements of the spaces are given directly, not through the mediation of a representation. However, these abstract spaces presuppose the real numbers. In this paper, we show how to set up an abstract type for the real numbers. The appropriateness of our construction works in tandem with the bounded functional interpretation.

An interpretation of **HA**…

An Interpretation of $\mathbf{HA}$

$$
\begin{aligned}
\{s =_{\mathbb{N}} t\} &:\equiv s = t \\
\{A \wedge B\}^{a,c}_{b,d} &:\equiv \{A\}^{a}_{b} \wedge \{B\}^{c}_{d} \\
\{A \to B\}^{f,g}_{a,d} &:\equiv \forall b \in gad\, \{A\}^{a}_{b} \to \{B\}^{fa}_{d} \\
\{\exists n^{\mathbb{N}} A\}^{a,m}_{B} &:\equiv \exists n \leq_{\mathbb{N}} m \forall b \in B \{A\}^{a}_{b} \\
\{\forall n^{\mathbb{N}} A\}^{f}_{b,m} &:\equiv \forall n \leq_{\mathbb{N}} m \{A\}^{fm}_{b}
\end{aligned}
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\end{aligned}}$$

$$A \quad \mapsto \quad \exists a \forall b \{A\}^a_b \qquad (\ \exists a \forall b\, A_I(a; b)\ )$$

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$$\forall k \forall n \leq k(tn = 0 \to \exists m \leq \phi k(snm = 0))$$

$$\frac{\forall n^{\mathbb{N}}(tn = 0 \rightarrow \exists m^{\mathbb{N}}(snm = 0))}{A}$$

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- Quantifiers need extra care
- Soundness of non-logical axioms easy to verify
- Interpreting induction already requires a ‘monotonicity’ property

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- We discuss monotonicity property when adding $A \vee B$
- Interpretation does not rely on majorizability (yet!)
- We use majorizability when interpreting $\mathbf{HA}^\omega$ into $\mathbf{HA}^\omega$

Interpreting disjunction…

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- Hence:

$$\{A \vee B\}^{a,c}_{B,D} \equiv \forall b \in B\{A\}^a_b \vee \forall d \in D\{B\}^c_d$$

An Interpretation (with disjunction)

$$\{s =_{\mathbb{N}} t\} \quad :\equiv s = t$$

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$$\{A \vee B\}^{a,c}_{B,D} :\equiv \forall b \in B \{A\}^a_b \vee \forall d \in D \{B\}^c_d$$

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$$\boxed{\begin{aligned} \mathrm{m}_{\mathbb{N}}(n_1, n_2) &:\equiv \max(n_1, n_2) \\ \mathrm{m}_{\rho*}(A_1, A_2) &:\equiv A_1 \cup A_2 \\ \mathrm{m}_{\rho\to\tau}(f_1, f_2) &:\equiv \lambda x \,.\, \mathrm{m}_\tau(f_1 x, f_2 x) \end{aligned}}$$

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- Monotonicity property:

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- Axioms for bounded quantifiers are interpretable:

$$\exists n \leq_{\mathbb{N}} m A \leftrightarrow \exists n^{\mathbb{N}} (n \leq_{\mathbb{N}} m \wedge A)$$
$$\forall n \leq_{\mathbb{N}} m A \leftrightarrow \forall n^{\mathbb{N}} (n \leq_{\mathbb{N}} m \rightarrow A)$$

Adding more sorts…

Adding abstract spaces, e.g. $\mathbb{R}$

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- Interpret relations between reals as

$$\{x =_{\mathbb{R}} y\}_k :\equiv |x - y| \leq 2^{-k}$$

$$\{x <_{\mathbb{R}} y\}^k :\equiv x + 2^{-k} < y$$

Soundness of disjunction $A \vee B$

- Define $\mathrm{m}_\tau(a_1, a_2)$ inductively as:

$$\boxed{\begin{aligned} \mathrm{m}_{\mathbb{N}}(n_1, n_2) &:\equiv \max(n_1, n_2) \\ \mathrm{m}_{\mathbb{R}}([p_1, q_1], [p_2, q_2]) &:\equiv [\min(p_1, p_2), \max(q_1, q_2)] \\ \mathrm{m}_{\rho^*}(A_1, A_2) &:\equiv A_1 \cup A_2 \\ \mathrm{m}_{\rho \to \tau}(f_1, f_2) &:\equiv \lambda x \,.\, \mathrm{m}_\tau(f_1 x, f_2 x) \end{aligned}}$$

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- We can ensure that $\mathbb{N}$ are the natural numbers in $\mathbb{R}$ by assuming $e \colon \mathbb{N} \to \mathbb{R}$ with axioms

$$e(0_{\mathbb{N}}) = 0_{\mathbb{R}} \qquad \forall n^{\mathbb{N}}(e(S_{\mathbb{N}}(n)) = S_{\mathbb{R}}(e(n)))$$

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$$\forall m, k \forall n \leq m \exists x \in \phi m(|e(n) - x| \leq 2^{-k})$$

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$$\forall x, y \in [p, q](x, y >_{\mathbb{R}} 2^{-k} \to \exists n \leq s2^k(nx >_{\mathbb{R}} y))$$

Inverse

- The following is also interpretable

$$\forall x^{\mathbb{R}}(x \neq_{\mathbb{R}} 0 \to \exists y^{\mathbb{R}}(xy = 1))$$

- In this case we are given that $x \in [p, q]$ and that $|x| \geq 2^{-k}$ for some k

- Hence, we can find an interval which bounds y in terms of p, q and k

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$$\forall x \in [p, q]\, \exists n^{\mathbb{N}} A(x, n) \to \exists N\, \forall x \in [p, q]\, \exists n \leq N\, A(x, n)$$

An interpretation of $\mathbf{HA}^{\omega}$…

Howard majorizability

- Lifts $n \leq_{\mathbb{N}} m$ to all types

$$n \leq^H_{\mathbb{N}} n^* :\equiv n \leq_{\mathbb{N}} n^*$$

$$f \leq^H_{\rho\to\tau} f^* :\equiv \forall x^* \forall x \leq^H_{\rho} x^* (fx \leq^H_{\tau} f^* x^*)$$

Quantifications over finite types

- We can lift the interpretation of $\mathbb{N}$

$$\{\exists n^{\mathbb{N}} A\}_{B}^{a,m} :\equiv \exists n \leq_{\mathbb{N}} m \forall b \in B\, \{A\}_{b}^{a}$$

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$$\{\exists x^{\tau} A\}_B^{a,x*} :\equiv \exists x \leq_{\tau}^{H} x* \forall b \in B \{A\}_b^a$$

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An Interpretation of $\mathbf{HA}^\omega$

$$
\begin{aligned}
\{s = t\} &:\equiv s = t \\
\{A \wedge B\,|^{a,c}_{b,d} &:\equiv \{A\}^a_b \wedge \{B\}^c_d \\
\{A \vee B\,|^{a,c}_{B,D} &:\equiv \forall b \in B\{A\}^a_b \vee \forall d \in D\{B\}^c_d \\
\{A \to B\,|^{f,g}_{a,d} &:\equiv \forall b \in gad\,\{A\}^a_b \to \{B\}^{fa}_d \\
\{\,\exists x^\tau A\,|^{a,x*}_{B} &:\equiv \exists x \leq^H_\tau x* \forall b \in B\{A\}^a_b \\
\{\,\forall x^\tau A\,|^{f}_{b,x*} &:\equiv \forall x \leq^H_\tau x*\{A\}^{fx*}_b
\end{aligned}
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Majorizability in finite types + $\mathbb{R}$

- Lift $n \leq_{\mathbb{N}} m$ and $x \in_{\mathbb{R}} [p, q]$ to all types

$$n \leq^H_{\mathbb{N}} m :\equiv n \leq_{\mathbb{N}} m$$

$$x \leq^H_{\mathbb{R}} [p, q] :\equiv x \in [p, q]$$

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Probability Space

Probability spaces $(\Omega, \mathscr{F}, P)$

- Sample space Ω can be an abstract space with uniform interpretation

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- Given the uniform interpretation of $\mathbb{B}$ the interpretation of $\Omega \to \mathbb{B}$ is also uniform:

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Work in progress…