On Uniform Interpretations of Quantifiers

A Uniform Realizability Interpretation

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Dedicated to Ulrich Berger on the occasion of his retirement

Swansea, 25 June 2025

Plan

- Uniform Interpretations of Quantifiers

  (a bit of history...)

- A Uniform Realizability Interpretation

  (parametrised by a base interpretation)

- Some Base Interpretations

  (examples of base interpretations)

A Thank You to Ulrich

Paulo Oliva
Visit to Swansea
To: U Berger

Berger 2 July 2001 at 22:35

Dear Ulrich,

I hope you have
Aarhus. And than

Here in Aarhus w
Kohlenbach. The
interesting cour
on Saturday and
very good speake
much his lecture

I have talked wi
that is a good i
September. Since
September I thin
My plan is to st

MODIFIED BAR RECURSION AND CLASSICAL DEPENDENT CHOICE

Abstract. We introd
"modified bar recursion"
choice allowing for the
As another application,
together with a version o
$\mathcal{M}$ of strongly majorizab

Modified Bar Recursion

Ulrich Berger[1] and Paulo Oliva[2]

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Received 23 November 2005

This paper studies modified bar recursion, a higher type recursion scheme which has been used in (BBC98) and (BO05) for a realizability interpretation of classical analysis. A complete clarification of its relation to Spector's and Kohlenbach's bar recursion, the fan functional, Gandy's functional Γ and Kleene's notion of S1-S9 computability is given.

A Thank You to Ulrich

Unifying Functional Interpretations

Paulo Oliva

Notre Dame Journal of Formal Logic 47 (2):263-290 (2006) Copy BibTeX

Abstract

This article presents a parametrized functional interpretation. Depending on the choice of two parameters one obtains well-known functional interpretations such as Gödel's Dialectica interpretation, Diller-Nahm's variant of the Dialectica interpretation, Kohlenbach's monotone interpretations, Kreisel's modified realizability, and Stein's family of functional interpretations. A functional interpretation consists of a formula interpretation and a soundness proof. I show that all these interpretations differ only on two design choices: first, on the number of counterexamples for A which became witnesses for ¬A when defining the formula interpretation and, second, the inductive information about the witnesses of A which is considered in the proof of soundness. Sufficient conditions on the parameters are also given which ensure the soundness of the resulting functional interpretation. The relation between the parametrized interpretation and the recent bounded functional interpretation is also discussed

Acknowledgments

Various insights which led to this work were gained during the development of the bounded functional interpretation, which I have carried out together with Fernando Ferreira. I am also grateful to Ulrich Berger for suggesting the possibility of a common framework for the different functional interpretations, to both Fernando Ferreira and Ulrich Kohlenbach, and to the anonymous referee for their valuable comments on earlier versions of this paper. The author gratefully acknowledges support of the UK EPSRC under grant GR/S31242/01.

A Thank You to Ulrich (and Monika)

Swansea, 2001

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Uniform Interpretations of Quantifiers

2003

UNIFORM HEYTING ARITHMETIC

ULRICH BERGER

Dedicated to Helmut Schwichtenberg on his 60th Birthday

Abstract. We present an extension of Heyting Arithmetic in finite types called *Uniform Heyting Arithmetic* (HA^{u}) that allows for the extraction of optimized programs from constructive and classical proofs. The system HA^{u} has two sorts of first-order quantifiers: ordinary quantifiers governed by the usual rules, and uniform quantifiers subject to stronger variable conditions expressing roughly that the quantified object is not computationally used in the proof. We combine a Kripke-style Friedman/Dragalin translation which is inspired by work of Coquand and Hofmann and a variant of the refined A-translation due to Buchholz, Schwichtenberg and the author to extract programs from a rather large class of classical first-order proofs while keeping explicit control over the levels of recursion and the decision procedures for predicates used in the extracted program.

$$r \,\mathbf{mr}\, \exists x^{\rho}\, A = \begin{cases} \mathsf{p}_1(r) \,\mathbf{mr}\, A[\mathsf{p}_0(r)/x] & \text{if } A \text{ is non-Harrop} \\ \epsilon \,\mathbf{mr}\, A[r/x] & \text{if } A \text{ is Harrop} \end{cases}$$

$$r \,\mathbf{mr}\, \mathsf{Q}A = \mathsf{Q}(r \,\mathbf{mr}\, A) \text{ where } \mathsf{Q} \in \{\{\forall x\}, \{\exists x\}\}$$

Uniform Interpretations of Quantifiers

Logical Methods in Computer Science
Volume 19, Issue 3, 2023, pp. 1:1–1:62
https://lmcs.episciences.org/

2003

Submitted Jun. 01, 2021
Published Jul. 06, 2023

COMPUTING WITH INFINITE OBJECTS: THE GRAY CODE CASE

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Program extraction is performed via a ‘uniform’ realisability interpretation (Section 4.4). Uniformity concerns the interpretation of quantifiers: A formula $(\forall x)\, A(x)$ is realised uniformly by one object a that realises $A(x)$ for all x, so a may not depend on x. Dually, a formula $(\exists x)\, A(x)$ is realised uniformly by one object a that realises $A(x)$ for some x, so a does not contain a witness for x. Expressions (formulas, predicates, operators) that contain

Uniform Interpretations of Quantifiers

Typed lambda-calculus in classical Zermelo-Frænkel set theory

2001

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In this paper, we develop a system of typed lambda-calculus for the Zermelo-Frænkel set theory, in the framework of classical logic. The first, and the simplest system of typed lambda-calculus is the *system of simple types*, which uses the intuitionistic propositional calculus, with the only connective →. It is very important, because the well known Curry-

The definition is given by induction on F :

$$|F \to G| = (|F| \to |G|) \; ; \; |\forall x\, F| = \bigcap_a |F[a/x]|.$$

Therefore :

$t \Vdash (F \to G)$ is the formula $(\forall u \in \Lambda)(u \Vdash F \to tu \Vdash G)$;

$t \Vdash \forall x\, F$ is the formula $\forall x(t \Vdash F)$.

Uniform Interpretations of Quantifiers

A functional interpretation for nonstandard arithmetic 2012

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A R T I C L E   I N F O

Article history:
Received 23 February 2012
Received in revised form 12 July 2012
Accepted 16 July 2012
Available online 3 August 2012
Communicated by J.M.E. Hyland

MSC:
03F10
03F50
11U10
26E35

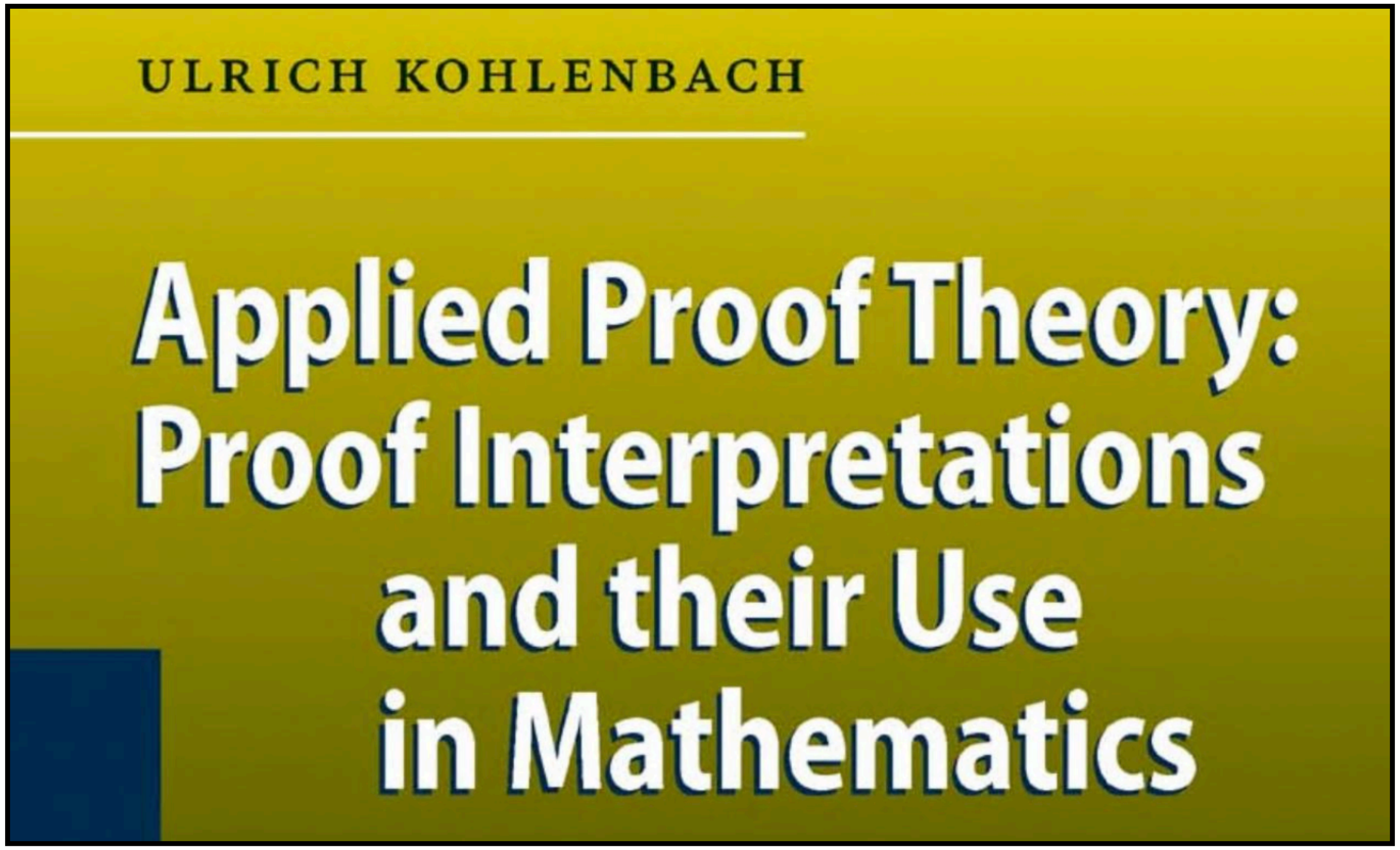
Keywords:
Proof theory
Functional interpretations
Nonstandard arithmetic

A B S T R A C T

We introduce constructive and classical systems for nonstandard arithmetic and show how variants of the functional interpretations due to Gödel and Shoenfield can be used to rewrite proofs performed in these systems into standard ones. These functional interpretations show in particular that our nonstandard systems are conservative extensions of E-HA$^\omega$ and E-PA$^\omega$, strengthening earlier results by Moerdijk and Palmgren, and Avigad and Helzner. We will also indicate how our rewriting algorithm can be used for term extraction purposes. To conclude the paper, we will point out some open problems and directions for future research, including some initial results on saturation principles.

$$
\begin{aligned}
\underline{s}\ \mathrm{hr}\, \exists x\, \Phi(x) &:\equiv \exists x\, (\underline{s}\, \mathrm{hr}\, \Phi(x)),\\
\underline{s}\ \mathrm{hr}\, \forall x\, \Phi(x) &:\equiv \forall x\, (\underline{s}\, \mathrm{hr}\, \Phi(x)),\\
s, \underline{t}\ \mathrm{hr}\, \exists^{\mathrm{st}} x\, \Phi(x) &:\equiv \exists s' \in s\, (\underline{t}\, \mathrm{hr}\, \Phi(s')),\\
\underline{s}\ \mathrm{hr}\, \forall^{\mathrm{st}} x\, \Phi(x) &:\equiv \forall^{\mathrm{st}} x\, (\underline{s}[x]\, \mathrm{hr}\, \Phi(x)).
\end{aligned}
$$

Uniformity in Proof Mining



This book gives an introduction to so-called proof interpretations, more specifically various forms of realizability and functional interpretations, and their use in mathematics. Whereas earlier treatments of these techniques (e.g. [362, 264, 121, 365, 7]) emphasize foundational and logical issues the focus of this book is on applications of the methods to extract new effective information such as computable uniform bounds from given (typically ineffective) proofs. This line of research, which has its roots in G. Kreisel's pioneering work on 'unwinding of proofs' from the 50's, has

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  (parametrised by a base interpretation)
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  (examples of base interpretations)

Heyting Arithmetic

Definition (Heyting arithmetic).

Assume $\mathbf{HA}$ formalised with three predicate symbols

- Falsity $\bot$ — nullary
- Natural number $\mathbb{N}(n)$ — unary
- Equality $n = m$ — binary

Notation.

$\forall n^{\mathbb{N}} A(n)$ is an abbreviation for $\forall n(\mathbb{N}(n) \to A(n))$

$\exists n^{\mathbb{N}} A(n)$ is an abbreviation for $\exists n(\mathbb{N}(n) \wedge A(n))$

A Uniform Realizability Interpretation

Notation.

Realizers live in some (typed) partial combinatory algebra $(A, \bullet)$.

Individuals live in some model $\mathscr{M}$

We use $\vec{x}$ for tuples of elements of $\mathscr{M}$ and $\boldsymbol{a}$ for tuples of realizers

Definition (Base Interpretation).

Associate to each n-ary predicate symbol P an $(n+m)$-ary relation

$$\vec{x} \lhd_P \boldsymbol{a}$$

between individuals and P-realizers (or P-bounds).

Example.

For the unary predicate $\mathbb{N}(n)$ we could take $n \lhd_{\mathbb{N}} \cdot$ to be:

$$n \lhd_{\mathbb{N}} m \; :\equiv \; n = m \quad \text{(precise)} \qquad\qquad n \lhd_{\mathbb{N}} \langle\,\rangle \; :\equiv \; \mathbb{N}(n) \quad \text{(uniform)}$$

A Uniform Realizability Interpretation

Definition (Uniform Realizability Interpretation).

Assume a fixed PCA $(A, \bullet)$ and base interpretation. Let:

$$
\begin{aligned}
\boldsymbol{a} \text{ ur } P(\vec{x}) &:\equiv \vec{x} \lhd_P \boldsymbol{a} \\
\boldsymbol{a}, \boldsymbol{b} \text{ ur } A \wedge B &:\equiv (\boldsymbol{a} \text{ ur } A) \wedge (\boldsymbol{b} \text{ ur } B) \\
\boldsymbol{f} \text{ ur } A \to B &:\equiv \forall \boldsymbol{a}((\boldsymbol{a} \text{ ur } A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } B)) \\
\boldsymbol{a} \text{ ur } \exists x A &:\equiv \exists x (\boldsymbol{a} \text{ ur } A) \\
\boldsymbol{a} \text{ ur } \forall x A &:\equiv \forall x (\boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

It follows that...

$$
\begin{aligned}
\boldsymbol{a}, \boldsymbol{b} \text{ ur } \exists n^{\mathbb{N}} A &:\equiv \exists n \lhd_{\mathbb{N}} \boldsymbol{a} \, (\boldsymbol{b} \text{ ur } A) \\
\boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &:\equiv \forall \boldsymbol{a} \forall n \lhd_{\mathbb{N}} \boldsymbol{a} \, (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

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Kleene Realizability

Definition (Kleene Base Interpretation).

Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\ n \lhd_{\mathbb{N}} m &:\equiv n = m \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

Take Kleene's first algebra $\mathscr{K}_1$ as the PCA.

It follows that...

$$\begin{aligned} n, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(n) \\ \boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n(\{\boldsymbol{a}\}(n)\downarrow \wedge \{\boldsymbol{a}\}(n) \text{ ur } A) \end{aligned}$$

Kreisel Modified Realizability

Definition (Kreisel Base Interpretation).

Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\ n \lhd_{\mathbb{N}} m^{\mathbb{N}} &:\equiv n = m \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

Take Gödel's system T as the typed PCA.

It follows that...

$$\begin{aligned} n^{\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(n) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ ur } A) \end{aligned}$$

Herbrand Realizability

Definition (Herbrand Base Interpretation).

Assume an extra unary predicate $\mathrm{std}(n)$ (for n is a **standard number**). Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\perp} \langle\,\rangle &:\equiv \perp \\ n \lhd_{\mathbb{N}} \langle\,\rangle &:\equiv \mathbb{N}(n) \\ n \lhd_{\mathrm{std}} S &:\equiv n \in S \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \exists n^{\mathbb{N}}(\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A(n) &\Leftrightarrow \forall n^{\mathbb{N}}(\boldsymbol{a} \text{ ur } A(n)) \\ S^{\mathbb{N}^*}, \boldsymbol{a} \text{ ur } \exists n^{\mathrm{std}} A(n) &\Leftrightarrow \exists n \in S\ (\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathrm{std}} A &\Leftrightarrow \forall S\, \forall n \in S\ (\boldsymbol{f}(S) \text{ ur } A) \end{aligned}$$

Classical (Berger) Modified Realizability

Definition (Classical Base Interpretation).

Fix unary atomic predicate $P_\perp(n)$. Let:

$$\begin{aligned} \langle\,\rangle \lhd_\perp n &:\equiv P_\perp(n) \\ n \lhd_{\mathbb{N}} m^{\mathbb{N}} &:\equiv n = m \\ (n, m) \lhd_= \langle\,\rangle &:\equiv n = m \end{aligned}$$

Take Gödel's system T as the typed PCA.

Remarks.

- Combination of modified realizability and Friedman's A-translation
- We are then able to realize $\neg\neg\exists n^{\mathbb{N}} P_\perp(n) \to \exists n^{\mathbb{N}} P_\perp(n)$
- Similar to Krivine's (classical) realizability

Aschieri-Berardi Learning Realizability

Definition (Aschieri-Berardi Base Interpretation).

Assume a set of states $\mathbf{S}$. Parametrised by an $s \in \mathbf{S}$, let:

$$\begin{aligned}
\langle\,\rangle \lhd_{\bot} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) \neq s \\
n \lhd_{\mathbb{N}} \alpha^{\mathbf{S}\to\mathbb{N}} &:\equiv n = \alpha(s) \\
(n, m) \lhd_{=} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) = s \to n = m
\end{aligned}$$

Take Gödel's system T (with $\mathbf{S}$ as an extra base type) as the typed PCA.

It follows that...

$$\begin{aligned}
\alpha^{\mathbf{S}\to\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(\alpha(s)) \\
\boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ ur } A)
\end{aligned}$$

The uniform functional interpretation with informative types

F. Ferreira and P. Oliva

Definition 2.3 ($\mathcal{U}$-interpretation of $\mathcal{L}_{\mathbf{S}}$ into $\mathcal{L}_{\mathbf{T}}$). *Let be given a base interpretation of $\mathcal{L}_{\mathbf{S}}$ into $\mathcal{L}_{\mathbf{T}}$. For each formula A of $\mathcal{L}_{\mathbf{S}}$, we define its $\mathcal{U}$-*interpretation $\langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}}$ *into $\mathcal{L}_{\mathbf{T}}$. The definition is by induction on the logical structure of A.*

For atomic formulas $R(t_1, \ldots, t_n)$, its $\mathcal{U}$-interpretation is defined as the given information relation $\langle R(t_1, \ldots, t_n) \rangle^{\boldsymbol{c}}_{\boldsymbol{d}}$. For $\perp$ we define

$$\langle \perp \rangle \quad :\equiv \quad \perp.$$

Assuming that A and B have $\mathcal{U}$-interpretations $\langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\langle B \rangle^{\boldsymbol{c}}_{\boldsymbol{d}}$, respectively, we define:

$$
\begin{aligned}
\langle A \wedge B \rangle^{\boldsymbol{a},\boldsymbol{c}}_{\boldsymbol{b},\boldsymbol{d}} &:\equiv \langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} \wedge \langle B \rangle^{\boldsymbol{c}}_{\boldsymbol{d}} \\
\langle A \to B \rangle^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{a},\boldsymbol{d}} &:\equiv \forall \boldsymbol{b} \in \boldsymbol{gad} \langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} \to \langle B \rangle^{\boldsymbol{fa}}_{\boldsymbol{d}} \\
\langle \forall x^{\sigma} A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} &:\equiv \forall x^{\sigma} \langle A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} \\
\langle \exists x^{\sigma} A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{B}} &:\equiv \exists x^{\sigma} \forall \boldsymbol{b} \in \boldsymbol{B} \, \langle A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{b}}.
\end{aligned}
$$

Summary

- Quantifiers are "naturally" uniform
- Qualified quantifications (e.g. $\exists n^{\mathbb{N}} A(n)$) carry computational content because of the qualifying predicate $\mathbb{N}(n)$
- Currently working with Fernando Ferreira on uniform functional interpretations:
  - New interpretations of function spaces $\rho \to \tau$
  - Functional interpretation of extensionality
  - Systematic treatment of bounded (uniform) quantifiers

Happy Retirement, Ulrich!

Uniform Interpretations of Quantifiers

Mathematical Intuitionism

Introduction to Proof Theory

A. G. Dragalin 1988

3. If a function algebraic model A of the language Ω is given, then, for any formula of Ω one can define its *value in the model.* The value $\|\varphi\|$ of a formula φ will be a certain form of the function pseudo-Boolean algebra,

1. A *function pseudo-Boolean algebra* is given by a triple $\langle B, \widehat{D}, F\rangle$, where B is a pseudo-Boolean algebra (the algebra of *truth values*) and $\widehat{D}$ is a two-place function with nonempty domain of definition and with values in the algebra B. The nonempty set $V = \{\pi \mid \exists q((\pi, q) \in \operatorname{Dom} \widehat{D})\}$ is called the set

$\|\varphi\|$ as the form $f_1 \wedge f_2$, $f_1 \vee f_2$, or $f_1 \supset f_2$.

4) If φ has the form $\forall x\psi(x)$ or $\exists x\eta(x)$, then we define $\|\varphi\| = \forall x\|\psi(x)\|$, or, respectively, $\|\varphi\| = \exists x\|\eta(x)\|$.

More precisely, if x is not a parameter of $\psi(x)$, then we set $\|\forall x\psi(x)\| =$