On Functional Interpretations and Applied Proof Theory

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Plan

- Motivation

  Proof Mining: What more can we learn from a proof?

- Tools I: Functional interpretations

  Compositionality and uniformity

- Tools II: Negative translations

  Selection functions and backtracking

- Classical arithmetic and analysis

  Induction, comprehension, choice and bar recursion

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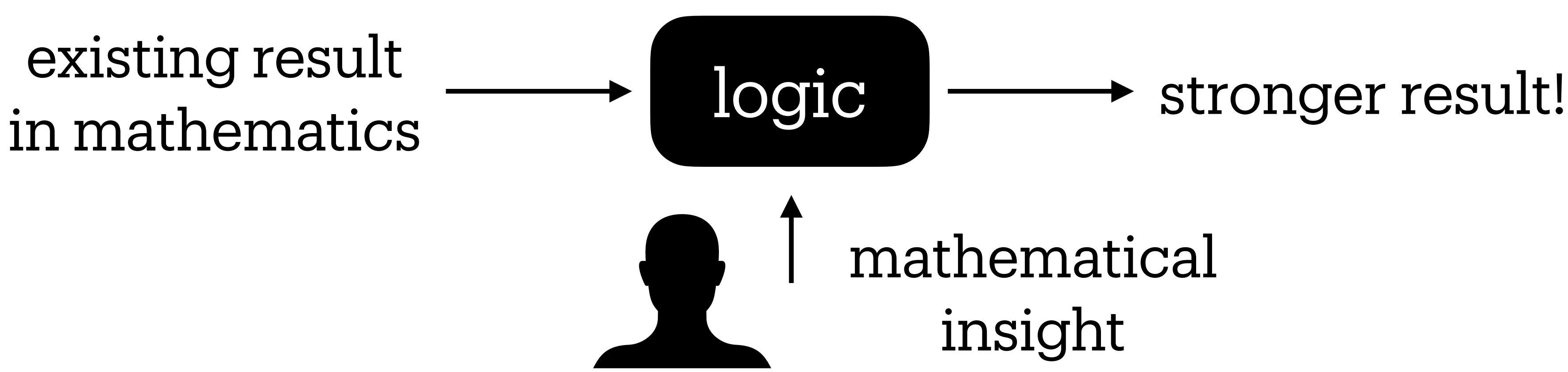
Unwinding of Proofs / Proof Mining

- Originated in 1950s with **Kreisel's** applications of his **no-counterexample interpretation**
- Resurgence from 1990s with **Kohlenbach's** applications of his **monotone functional interpretation**

Proof Mining (Kohlenbach).

Proof mining is the process of logically analysing (ineffective) proofs in mathematics with the aim of obtaining **new information**.



$$\sqrt{2} \notin \mathbb{Q}$$

$$\sqrt{2} \notin \mathbb{Q}$$

$$\forall p, q^{\mathbb{N}} \left(\frac{p}{q} \neq \sqrt{2} \right)$$

$$\forall p, q^{\mathbb{N}} \exists \varepsilon > 0 \left(\left| \frac{p}{q} - \sqrt{2} \right| \geq \varepsilon \right)$$

Theorem.

$\sqrt{2} \notin \mathbb{Q}$

Proof. (qualitative/soft mathematics)

Suppose (1) p, q coprimes

Suppose (2) $\sqrt{2} = p/q$

By (2), $2q^2 = p^2$, so $p = 2p'$ (even)

Hence, $2q^2 = 4(p')^2$

So, $q = 2q'$ (also even)

By (1), a contradiction. $\square$

Theorem.

$|\sqrt{2} - p/q| \geq 1/4q^2$

Proof. (quantitative/hard mathematics)

Suppose (1) $2q > p > q$ not both even

Then $2q^2 \neq p^2$ (see qualitative proof)

So $|2q^2 - p^2| \geq 1$

Since (easy to derive)

$(\dagger)$ $|a^2 - b^2| \geq \delta \rightarrow |a - b| \geq \delta/(a+b)$

we get

$$|\sqrt{2}q - p| \geq 1/(\sqrt{2}q + p) \geq 1/4q$$

That concludes the proof. $\square$

Theorem.

$\sqrt{2} \notin \mathbb{Q}$

Theorem.

$|\sqrt{2} - p/q| \geq 1/4q^2$

Proof. (qualitative/soft mathematics)

$$\dfrac{\dfrac{p,q\text{ coprime}\qquad \dfrac{\dfrac{\begin{array}{c}[\sqrt{2}=p/q]_\alpha\\ 2q^2=p^2\end{array}}{\cdots}}{p,q\text{ even}}}{\perp}}{\sqrt{2}\neq p/q}\;(\alpha)$$

Proof. (quantitative/hard mathematics)

$$\dfrac{\dfrac{p,q\text{ not both even}}{\cdot\,\cdot}}{2q^2\neq p^2}$$

$$\dfrac{|2q^2-p^2|\geq 1}{|\sqrt{2}-p/q|\geq 1/4q^2}\;(\star)$$

$(\star)$ new mathematical insight

Proof Mining

- [1992] Chebyshev Approximation (Kohlenbach)
- [2001] L_1 Approximation (Kohlenbach/O.)
- [2000s] Fixed-point Theory
  (Kohlenbach, Lambov, Gerhardy, Briseid, Leuștean,...)
- [2010s] Ergodic Theory
  (Avigad, Gerhardy, Towsner, Kohlenbach, Leuștean, Safarik,...)
- [2010s] Combinatorics (Kreuzer)
- [2010-2020s] Fixed-point Theory
  (Kohlenbach, Ferreira, Leuștean, Sipoş, Nicolae, Dinis, Pinto,...)
- [2020s] Probability Theory (Arthan, O., Pischke, Neri, Powell,...)

https://sites.google.com/view/proofmining

https://nicholaspischke.github.io/bib/ref_date.html

"Completing" Statements

$\sqrt{2}$ not rational

$$\forall p, q^{\mathbb{N}} \left(\sqrt{2} \neq p/q\right)$$

Modulus of distance Δ

$$\forall p, q^{\mathbb{N}} \left(|\sqrt{2} - p/q| \geq \Delta(p, q) \right)$$

Uniform continuity

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \; (|x - y| < \delta \to |f(x) - f(y)| < \varepsilon)$$

Modulus of uniform continuity Φ

$$\forall \varepsilon > 0 \forall x, y \; (|x - y| < \Phi(\varepsilon) \to |f(x) - f(y)| < \varepsilon)$$

Convergence

$$\forall \varepsilon > 0 \exists n \forall i \geq n \; (|x_i - c| < \varepsilon)$$

Modulus of convergence Ψ

$$\forall \varepsilon > 0 \forall i \geq \Psi(\varepsilon) \; (|x_i - c| < \varepsilon)$$

Uniqueness of solution

$$\forall x_1, x_2 \; (f(x_1) = 0 \wedge f(x_2) = 0 \to x_1 = x_2)$$

Modulus of uniqueness Θ

$$\forall x_1, x_2 \forall \varepsilon > 0 \; (|f(x_1)|, |f(x_2)| < \Theta(\varepsilon) \to |x_1 - x_2| < \varepsilon)$$

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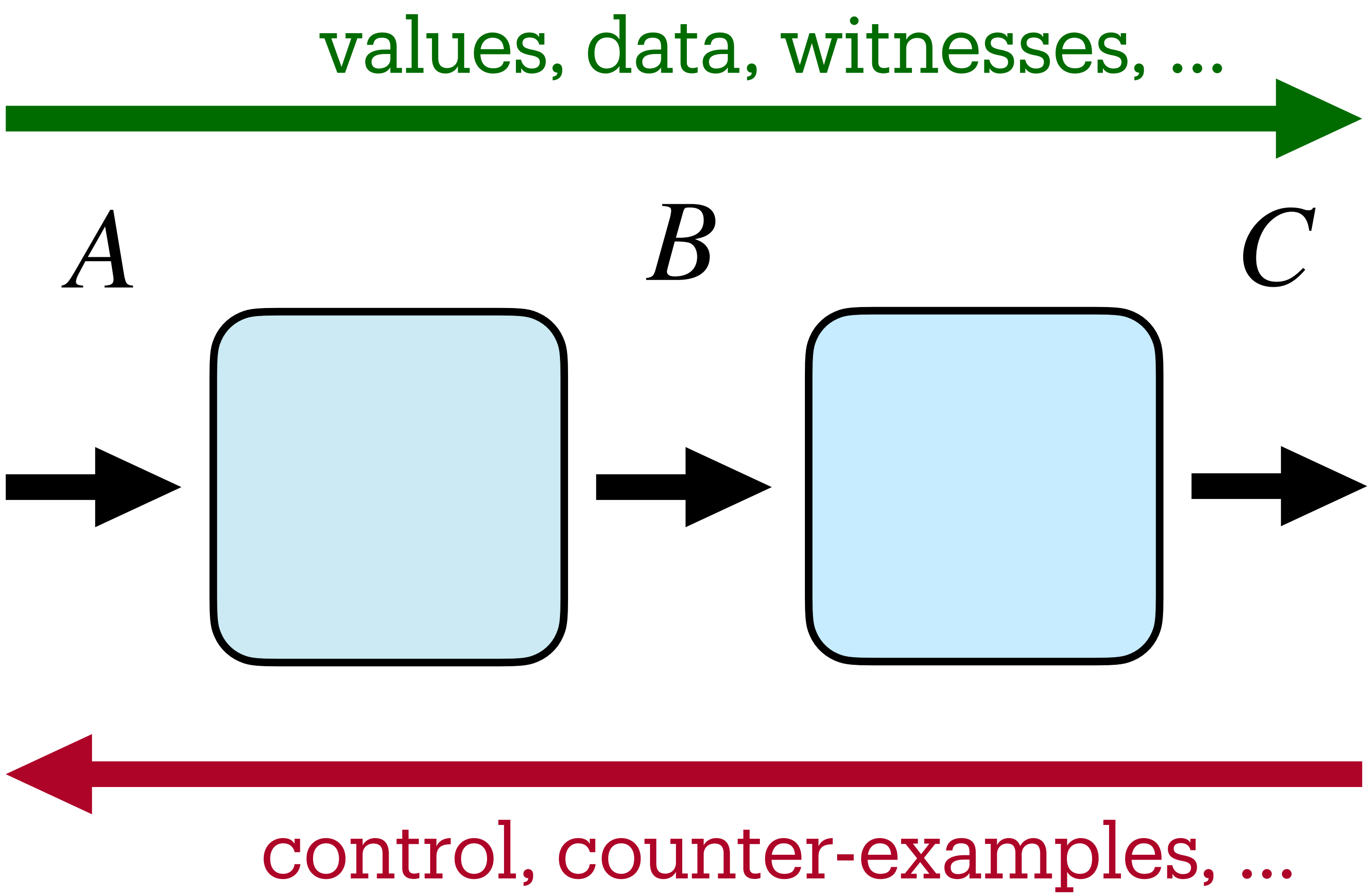
Functional Interpretations

Turn logical dependence into functional dependence



Functional Interpretations

Definition.

Functional interpretations associate formulas A to triples $(A^+, A^-, |A| \subseteq A^+ \times A^-)$

- A^+ is the set/type of evidence for A
- A^- is the set/type of counter-evidence for A
- $|A|^x_y$ determines whether evidence $x \in A^+$ wins against counter-evidence $y \in A^-$

Games.

Think of $(A^+, A^-, |A|^x_y)$ as a **game**

- A^+ strategies for Eloise
- A^- strategies for Abelard
- $|A|^x_y$ who wins, for $x \in A^+, y \in A^-$

Example ($\sqrt{2} \notin \mathbb{Q}$ game).

$A \equiv \forall p, q^{\mathbb{N}} \exists \varepsilon > 0(|p/q - \sqrt{2}| \geq \varepsilon)$

- $A^+ \equiv \mathbb{N}^2 \to \mathbb{Q}^+$
- $A^- \equiv \mathbb{N}^2$
- $|A|^f_{p,q} \equiv |p/q - \sqrt{2}| \geq f(p,q)$

Functional Interpretations

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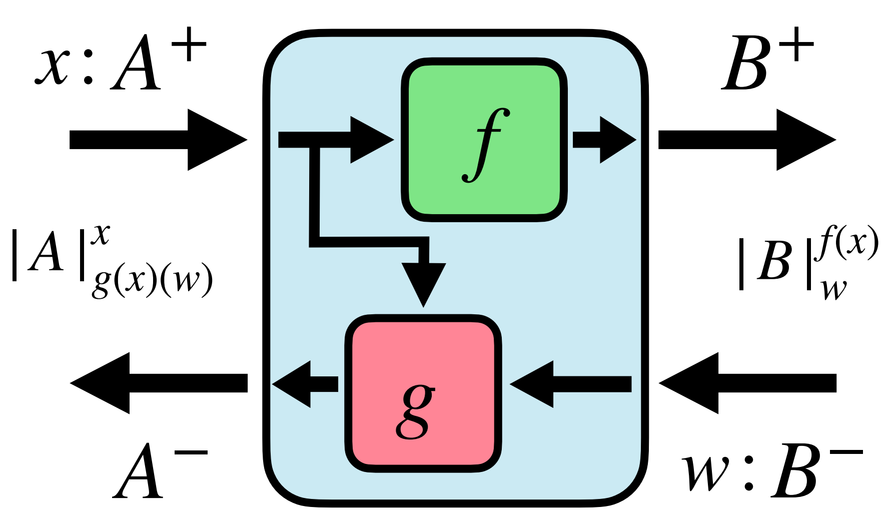
- A^+ is the set/type of evidence for A
- A^- is the set/type of counter-evidence for A
- $|A|^x_y$ determines whether evidence $x \in A^+$ wins against counter-evidence $y \in A^-$

Definition (Interpreting implication).

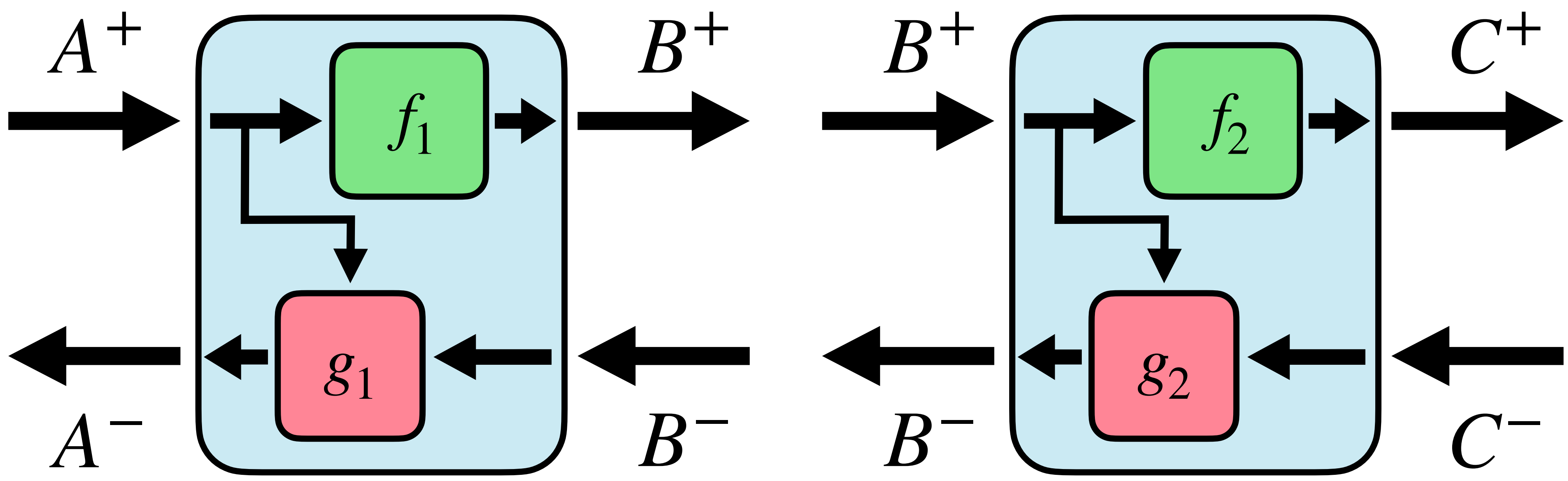
Implication $A \to B$ is interpreted as a pair of maps:

- $f \colon A^+ \to B^+$ (evidence map)
- $g \colon A^+ \to B^- \to A^-$ (counter-evidence map)
- for $x \in A^+$ and $w \in B^-$, we must have

$$|A|^x_{g(x)(w)} \to |B|^{f(x)}_w$$

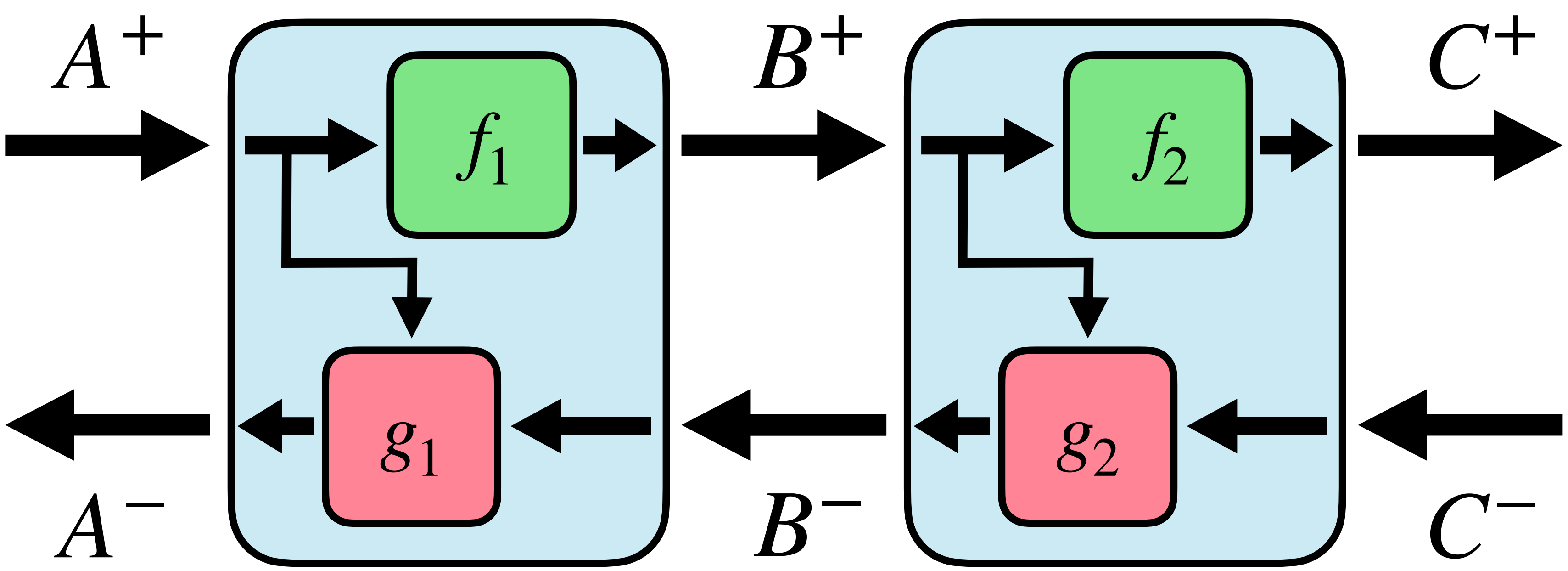


Composition



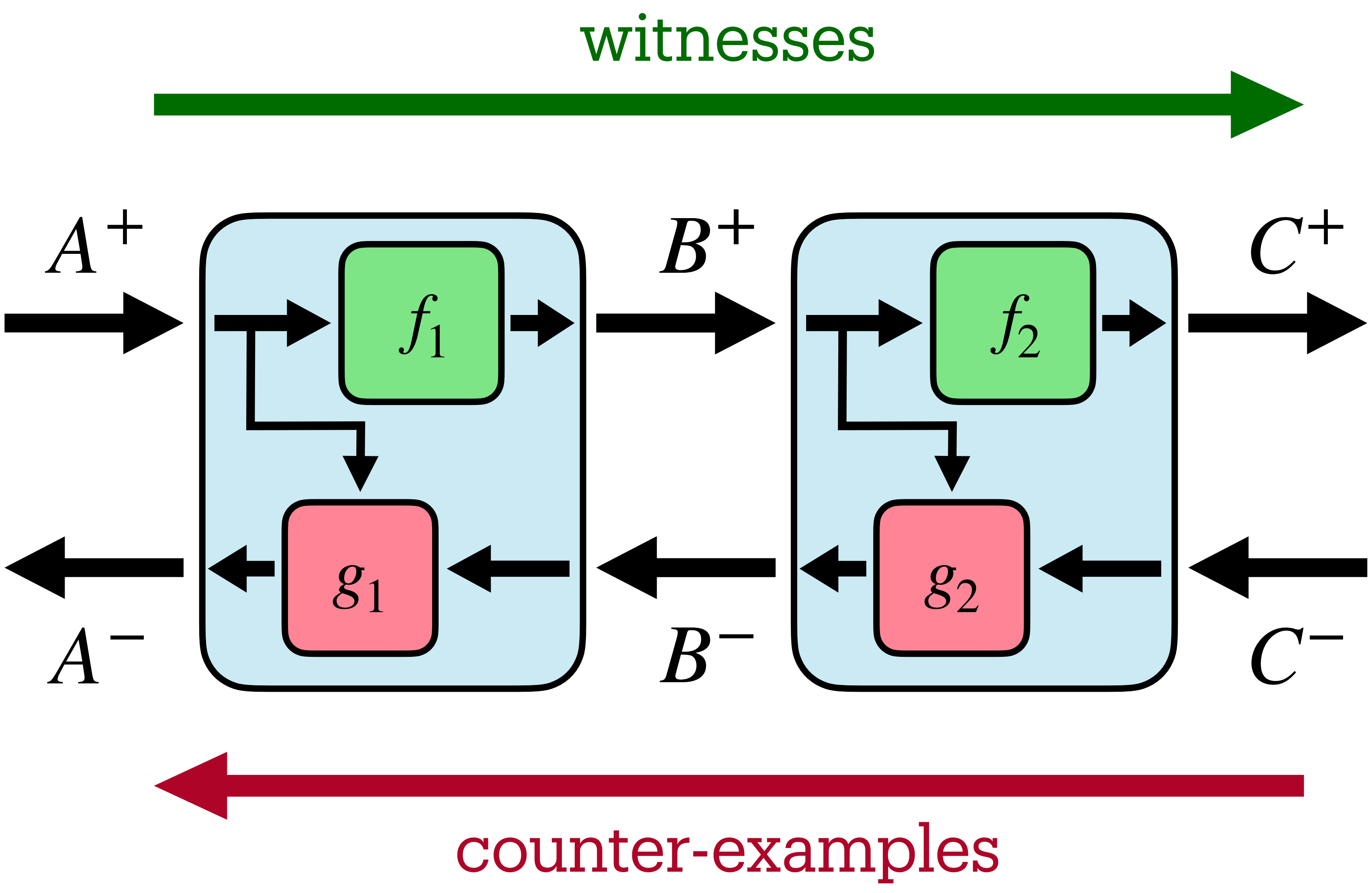
Composition

$$f_2 \circ f_1 : A^+ \to C^+$$

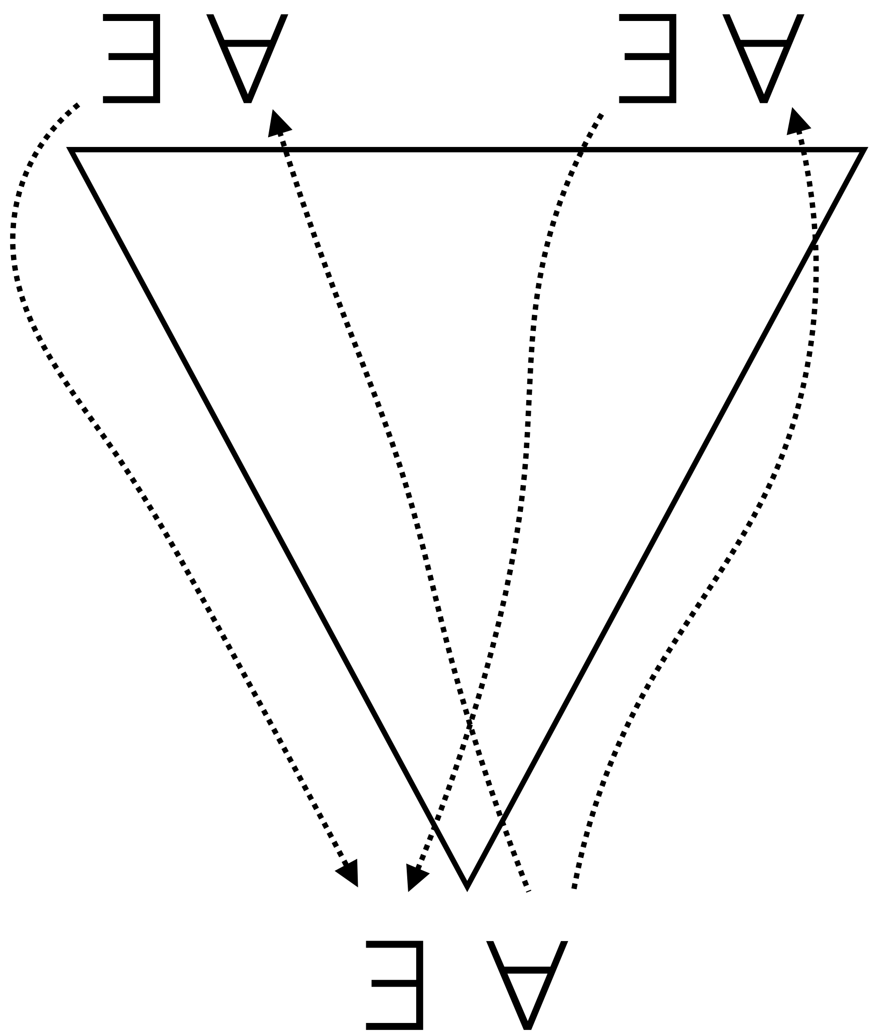


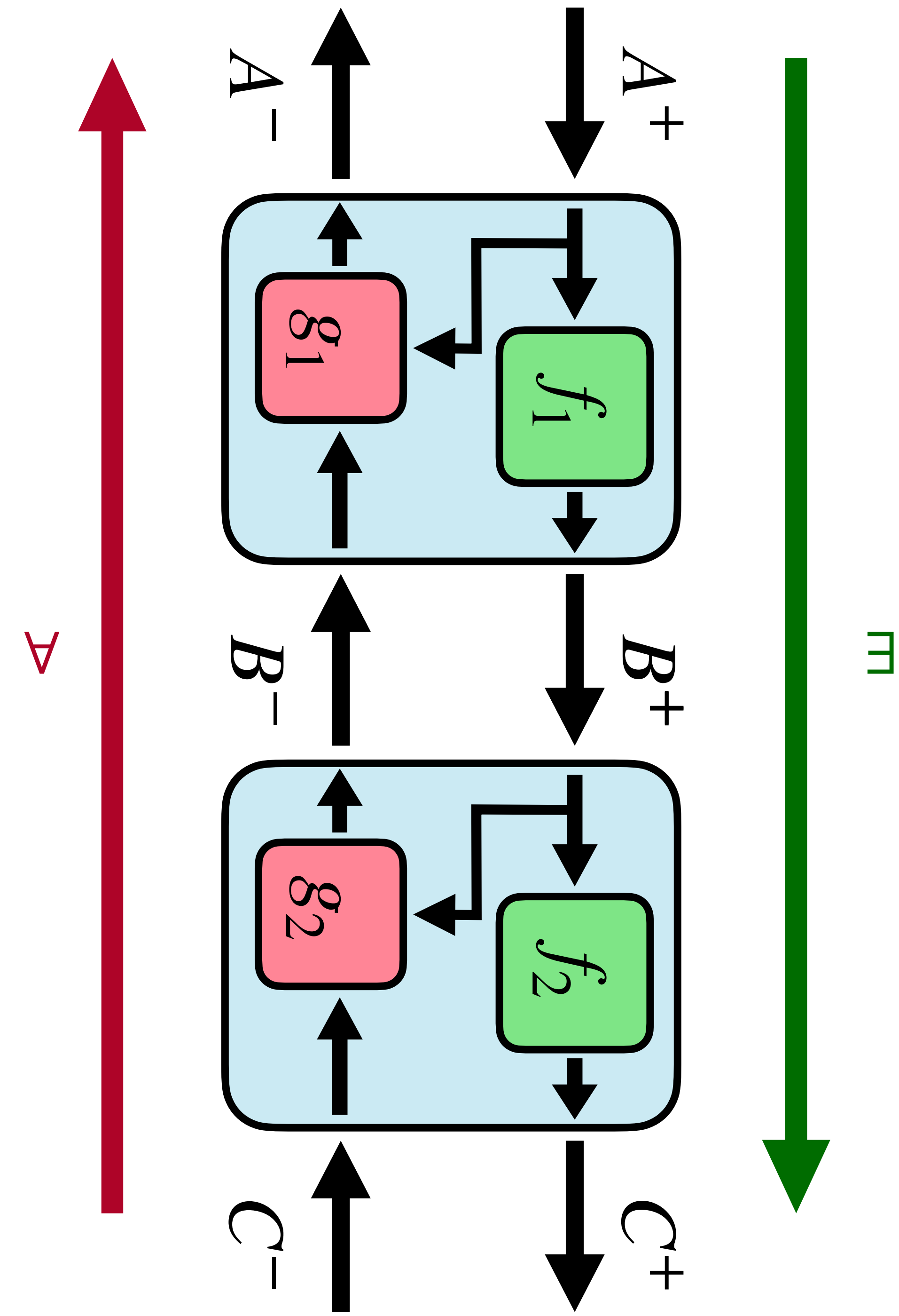
$$\lambda x^{A^+} . g_2(f_1(x)) \circ g_1(x) : A^+ \to C^- \to A^-$$

Logic

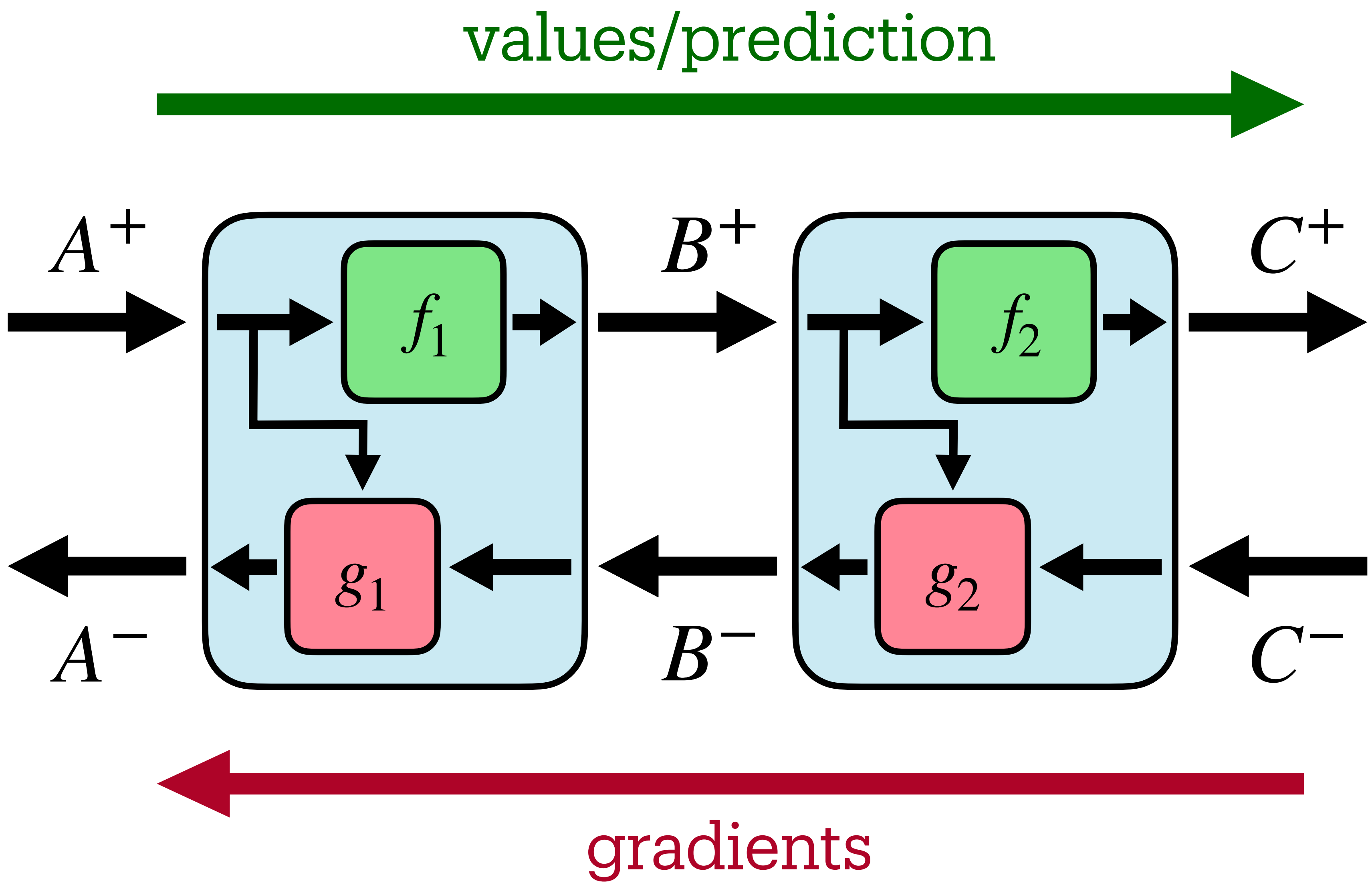


Functional Interpretations





Machine Learning (Neural Nets)



Kohlenbach Monotone Interpretations

Definition (Howard'73).

For each finite type τ define $x \leq^*_\tau a$, read "a **majorizes** x", inductively as

$$n \leq^*_{\mathbb{N}} m \quad :\equiv \quad n \leq m$$
$$f \leq^*_{\rho \to \tau} g \quad :\equiv \quad \forall a \forall x \leq^*_\rho a (f(x) \leq^*_\tau g(a))$$

Definition (Kohlenbach'96).

Given interpretation $(A^+, A^-, |A|^x_y)$ for A, its **monotone realizer** is an a such that

$$\exists x \leq^*_{A^+} a \forall y |A|^x_y$$

Key Idea.

If $A(v)$ but $v \leq^* t$ then monotone realizer is **uniform** on v (only depends on bound t)

Functional Interpretation of Linear Logic

Definition (de Paiva'1989, Shirahata'2006).

Suppose each atomic P of $\mathbf{LL}$ is associated with a triple $(P^+, P^-, |P| \subseteq P^+ \times P^-)$.

This can be extended to all formulas as

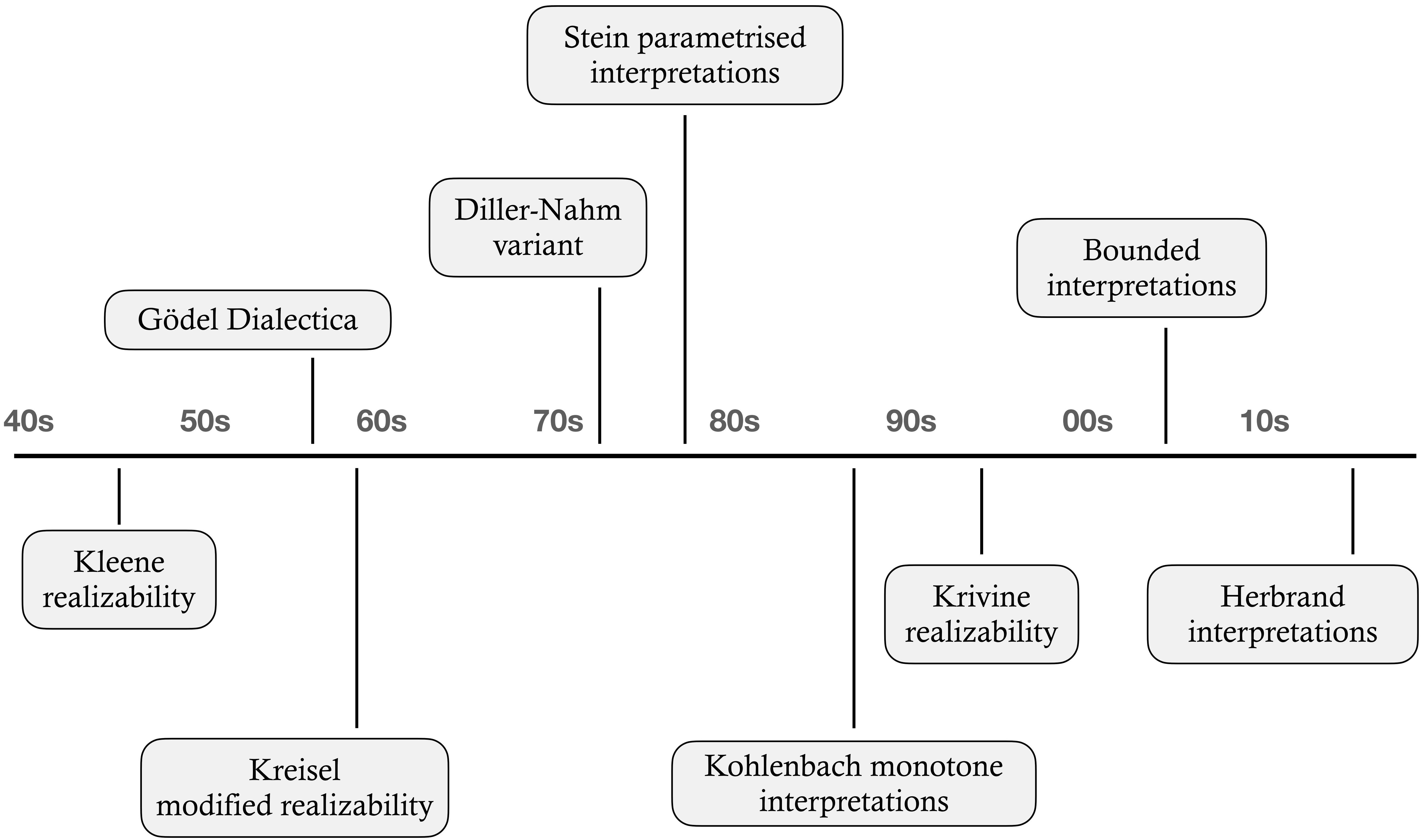
$$
\begin{array}{rcl}
|A \otimes B|^{x,v}_{f,g} & :\equiv & |A|^x_{g(v)} \text{ and } |B|^v_{f(x)} \\
|A \oplus B|^{b,x,v}_{y,w} & :\equiv & |A|^x_y \text{ if } b \text{ else } |B|^v_w \\
|A \multimap B|^{f,g}_{x,w} & :\equiv & |A|^x_{g(w)} \text{ implies } |B|^{f(x)}_w
\end{array}
\qquad
\begin{array}{rcl}
|A^\perp|^y_x & :\equiv & \text{not } |A|^x_y \\
|\forall z A(z)|^f_{v,y} & :\equiv & |A(v)|^{f(v)}_y \\
|!A|^x_g & :\equiv & \forall y \in g(x)\, |A|^x_y
\end{array}
$$

Definition (Other interpretations — see next slide).

$$
\begin{array}{rcl}
|!A|^x_g & :\equiv & |A|^x_{g(x)} \\
|!A|^x_g & :\equiv & \forall y \in g(x)\, |A|^x_y \\
|!A|^x & :\equiv & \forall y\, |A|^x_y
\end{array}
\qquad
\begin{array}{rcl}
|!A|^x_g & :\equiv & \forall y \leq^* g(x)\, |A|^x_y \\
|!A|^x_g & :\equiv & \forall y \in g(x)\, |A|^x_y \wedge A \\
|!A|^x & :\equiv & \forall y\, |A|^x_y \wedge A
\end{array}
$$

Valéria de Paiva, A Dialectica-like model of linear logic, 1989

Masaru Shirahata, The Dialectica interpretation of first-order classical linear logic, 2006


Stein parametrised interpretations
Diller-Nahm variant
Bounded interpretations
Gödel Dialectica
40s
50s
60s
70s
80s
90s
00s
10s
Kleene realizability
Krivine realizability
Herbrand interpretations
Kreisel modified realizability
Kohlenbach monotone interpretations

Unifying Functional Interpretations

- Parametrised interpretation of IL (2006)

  P. O., Unifying functional interpretations, NDJFL, 2006

- Parametrised interpretation of CLL and ILL (2007-2010)

  P. O., Modified realizability interpretation of classical linear logic, LICS, 2007

  P. O., Computational interpretations of classical linear logic, WoLLiC, 2007

  Gilda Ferreira & P. O., Functional interpretations of intuitionistic linear logic, CSL, 2009

  P. O., Functional interpretations of intuitionistic linear logic, Information & Computation, 2010

- Hybrid functional interpretations (2008-2012)

  Mircea-Dan Hernest & P. O., Hybrid functional interpretations, CiE, 2008

  P. O., Hybrid functional interpretations of linear and intuitionistic logic, JLC, 2012

- Interpretations with truth (2010)

  Jaime Gaspar & P. O., Proof interpretations with truth, MLQ, 2010

1. How to interpret atomic formulas

Two degrees of freedom...

2. How to interpret contraction

$|\mathbb{N}(n)|^a$ / $|\,!A\,|^a_b$

	$\|A\|^a_b$	$\forall b \in B\,\|A\|^a_b$	$\forall b \leq^* B\,\|A\|^a_b$	$\forall b\,\|A\|^a_b$
$\mathbb{N}(n)$	✗	**Herbrand f.i. (internal)**	💡	**Herbrand mr. (internal)**
$n \leq a$	💡	💡	**bounded f.i.**	**bounded mr**
$n \in a$	💡	**Herbrand f.i. (external)**	💡	**Herbrand real. (external)**
$n = a$	**Dialectica**	**Diller-Nahm**	✗	**modified realizability**

Bruno Dinis & P. O., A parametrised functional interpretation of Heyting arithmetic, 2021

Fernando Ferreira & P. O., The uniform functional interpretation with informative types, in preparation

Functional Interpretations: Uniformity

Definition (Computational interpretation of quantifiers).

Assume $A(z^\tau)$ has interpretation $(A^+, A^-, |A| \subseteq A^+ \times A^-)$.

Let $(\exists z^\tau A)^+ :\equiv \tau \times A^+$ and $(\exists z^\tau A)^- :\equiv A^-$ and

$$|\exists z^\tau A(z)|^{v,x}_y \;:\equiv\; |A(v)|^x_y$$

Let $(\forall z^\tau A)^+ :\equiv \tau \to A^+$ and $(\forall z^\tau A)^- :\equiv \tau \times A^-$ and

$$\boxed{|\forall z^\tau A(z)|^f_{v,y} \;:\equiv\; |A(v)|^{f(v)}_y}$$

Definition (Uniform interpretation of quantifiers).

Let $(\exists z^\tau A)^+ :\equiv A^+$ and $(\exists z^\tau A)^- :\equiv (A^-)^*$ and

$$|\exists z^\tau A(z)|^x_S \;:\equiv\; \exists z^\tau \forall y \in S\, |A(z)|^x_y$$

Let $(\forall z^\tau A)^+ :\equiv A^+$ and $(\forall z^\tau A)^- :\equiv A^-$ and

$$\boxed{|\forall z^\tau A(z)|^x_y \;:\equiv\; \forall z^\tau |A(z)|^x_y}$$

Fernando Ferreira & P. O., The uniform functional interpretation with informative types, in preparation

Uniform Functional Interpretation

Idea (F. Ferreira & P. O., work in progress).

1. Treat all quantifiers uniformly
2. Add new atomic formulas $I_\tau(x^\tau)$
3. Control interpretation of quantifications over x^τ via the interpretation of $I_\tau(x^\tau)$

$$|I_\tau(x^\tau)|^a \quad :\equiv \quad x = a \quad \text{(precise)} \qquad \text{or} \qquad |I_\tau(x^\tau)| \quad :\equiv \quad \text{true} \quad \text{(uniform)}$$

4. Function types $\rho \to \tau$ can be given new interpretations, including *canonical* one:

$$|I_{\rho\to\tau}(\phi)|^{f,g}_{x,w} \;:\equiv\; \forall z^\rho(\forall y \in g(x,w)\,|I_\rho(z)|^x_y \to |I_\tau(\phi(z))|^{f(x)}_w)$$

5. Restricted (bounded) quantifiers $\forall x^\tau(R(x) \to A(x))$ can be treated *uniformly* if

$$\forall x^\tau(R(x) \to I_\tau(x))$$

is uniformly interpretable

Fernando Ferreira & P. O., The uniform functional interpretation with informative types, in preparation

Uniform Functional Interpretation

Proposition (F. Ferreira & P. O., work in progress).

With **precise** interpretation of $I_{\mathbb{N}}(n)$ and **canonical** interpretation at function types

$$|\,I_{\rho\to\tau}(\phi)\,|^{f,g}_{x,w} \;\equiv\; \forall z^{\rho}(\forall y \in g(x,w)\,|\,I_{\rho}(z)\,|^{x}_{y} \to |\,I_{\tau}(\phi(z))\,|^{f(x)}_{w})$$

the **extensionality axiom** for type 2 functionals becomes interpretable:

$$\forall F^2, \alpha, \beta^1(\alpha =_1 \beta \to F(\alpha) =_0 F(\beta))$$

Proposition (F. Ferreira & P. O., work in progress).

With the **bounding** interpretation of $I_{\mathbb{N}}(n)$ and **canonical** interpretation at function types, quantifications $\forall \alpha \leq_1 t\, A(\alpha)$ and $\exists \alpha \leq_1 t\, A(\alpha)$ can be treated uniformly

$$|\forall \alpha \leq_1 t\, A(\alpha)\,|^{x}_{y} \;\equiv\; \forall \alpha \leq_1 t\,|\,A(\alpha)\,|^{x}_{y} \qquad |\,\exists \alpha \leq_1 t\, A(\alpha)\,|^{x}_{S} \;\equiv\; \exists \alpha \leq_1 t \forall y \in S\,|\,A(\alpha)\,|^{x}_{y}$$

Fernando Ferreira & P. O., The uniform functional interpretation with informative types, in preparation

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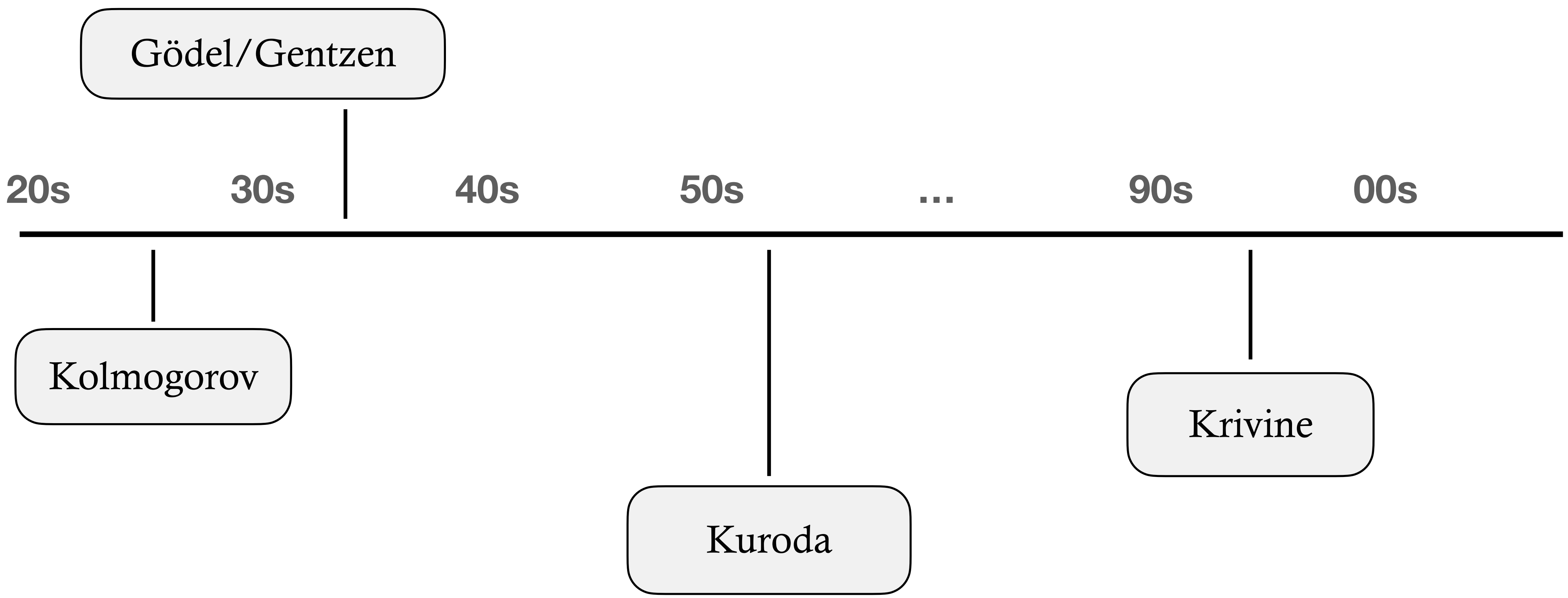
Negative (Double Negation) Translations



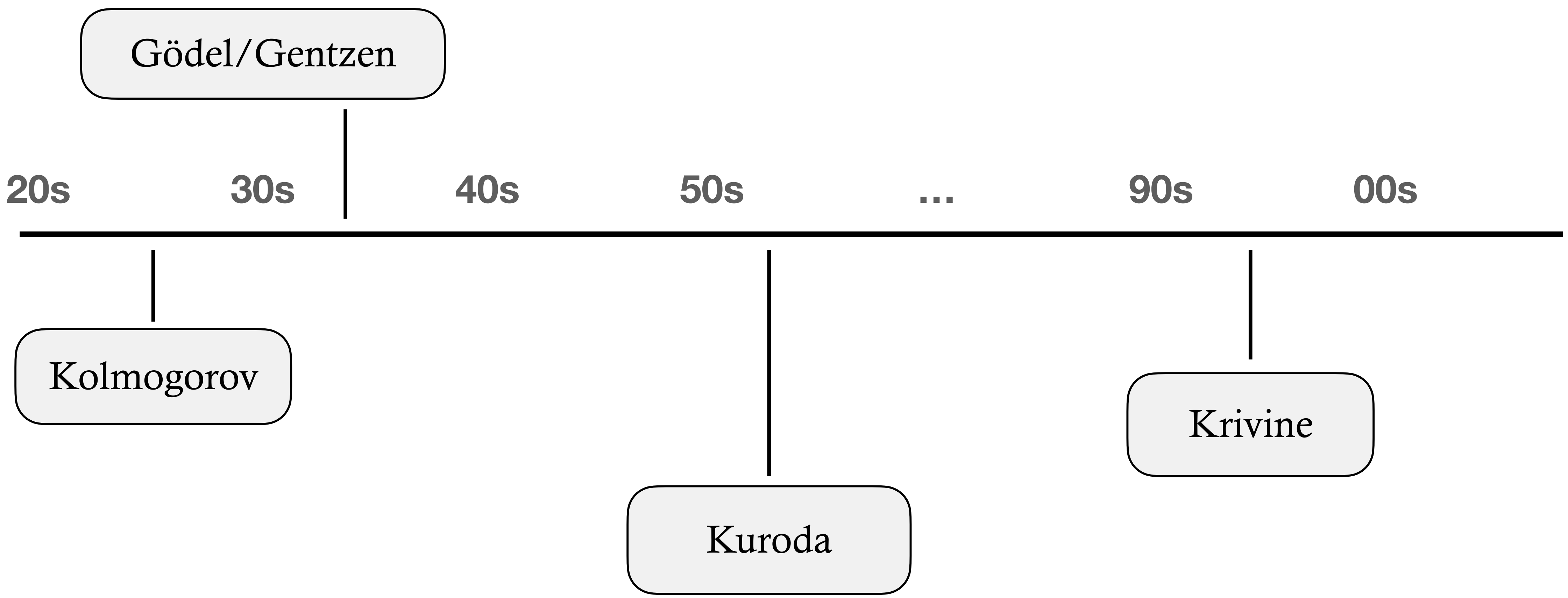
Excluded Middle "Tamed"

$$\boxed{\vdash A \vee \neg A} \qquad\qquad \boxed{\vdash \neg\neg(A \vee \neg A)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\bot}}{\neg A}\,\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\bot}}{A \vee \neg A}\ \text{PBC}, \gamma$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]_\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\bot}}{\neg A}\,\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\bot}}{\neg\neg(A \vee \neg A)}\ \to\mathrm{I}, \gamma$$

Negative (Double Negation) Translations



Theorem (Gödel/Gentzen 1933).

If **CL** proves $\Gamma \vdash A$ then **IL** proves $\Gamma^G \vdash A^G$

$$
\begin{array}{rcl@{\qquad}rcl}
(A \wedge B)^G & \equiv & A^G \wedge B^G & (P(\vec{x}))^G & \equiv & \neg\neg P(\vec{x}) \\
(A \vee B)^G & \equiv & \neg\neg(A^G \vee B^G) & (\forall x A)^G & \equiv & \forall x A^G \\
(A \to B)^G & \equiv & A^G \to B^G & (\exists x A)^G & \equiv & \neg\neg\exists x A^G
\end{array}
$$

Theorem (Kuroda 1951).

If **CL** proves $\Gamma \vdash A$ then **IL** proves $\Gamma^K \vdash \neg\neg A^K$

$$
\begin{array}{rcl@{\qquad}rcl}
(A \wedge B)^K & \equiv & A^K \wedge B^K & (P(\vec{x}))^K & \equiv & P(\vec{x}) \\
(A \vee B)^K & \equiv & A^K \vee B^K & (\forall x A)^K & \equiv & \forall x \neg\neg A^K \\
(A \to B)^K & \equiv & A^K \to B^K & (\exists x A)^K & \equiv & \exists x A^K
\end{array}
$$

Theorem (Ferreira/O'2012).

Gödel/Gentzen and Kuroda can be seen as **systematic simplifications** of the Kolmogorov translation ("from outside" and "from inside", respectively).

Gilda Ferreira & P. O., On the relation between various negative translations, 2012

Excluded Middle: Backtracking

$$\boxed{\vdash A \vee \neg A}$$

- Backtracking (learning) interpretation:
  - Claim $A \to \perp$
  - If claim gets used, with some A, to get contradiction ($\perp$)
  - Backtrack, claim A (using given A)

Thierry Coquand, Computational Content of Classical, 2009

Stefano Berardi, Thierry Coquand & Susumu Hayashi, Games with 1-backtracking, 2010

Functional interpretation of $\neg\neg A$

Observation.

If A has interpretation $(A^+, A^-, |A|^x_y)$ then

$$(\neg A)^+ \equiv A^+ \to A^- \qquad (\neg A)^- \equiv A^+ \qquad |\neg A|^f_x \equiv \neg |A|^x_{f(x)}$$

and

$$(\neg\neg A)^+ \equiv (A^+ \to A^-) \to A^+ \qquad (\neg\neg A)^- \equiv A^+ \to A^- \qquad |\neg\neg A|^\varepsilon_f \equiv \neg\neg |A|^{\varepsilon(f)}_{f(\varepsilon(f))}$$

Definition (Martín Escardó & P. O.).

We call a functional $\varepsilon \colon (X \to R) \to X$ a **selection function.**

For any (fixed) R the type mapping $J_R(X) = (X \to R) \to X$ is a **(strong) monad**.

The interpretation of $\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$ is given by the **product of selection functions** $\otimes \colon J_R(X) \times J_R(Y) \to J_R(X \times Y)$

Existence of certain selection function for X means X a "compact" types (Escardó)

Martín Escardó & P. O., Computational interpretations of analysis via products of selection functions, 2010

$\neg\neg(\exists n P(n) \vee \forall n \neg P(n))$

Example.

$A = \neg\neg(\exists n P(n) \vee \forall m \neg P(m))$ has (precise) interpretation:

$$A^+ \;\equiv\; (\mathbb{B} \times \mathbb{N} \to \mathbb{N}) \to \mathbb{B} \times \mathbb{N} \qquad A^- \;\equiv\; \mathbb{B} \times \mathbb{N} \to \mathbb{N}$$

and

$$|A|^{\varepsilon}_{f} \;\equiv\; \begin{cases} P(\varepsilon_1(f)) & \varepsilon_0(f) \\ \neg P(\varepsilon_1(f)) & \neg\varepsilon_0(f) \end{cases}$$

Indeed, we can take

$$\varepsilon(f^{\mathbb{B}\times\mathbb{N}\to\mathbb{N}}) \overset{\mathbb{B}\times\mathbb{N}}{=} \begin{cases} (\text{true}, f(\text{false},0)) & P(f(\text{false},0)) \\ (\text{false},0) & \neg P(f(\text{false},0)) \end{cases}$$

Intuition.

Claim $\forall m \neg P(m)$, i.e. false (witness 0 no relevant)

If $m = f(\text{false},0)$ is such that $P(m)$ we backtrack to $\exists n P(n)$ — i.e. (true, m)

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Induction

- Assume $A^- \equiv B^- \equiv R$
- The finite "double negation shift"

$$\neg\neg A \wedge \neg\neg B \to \neg\neg(A \wedge B)$$

is interpreted by the **binary product of selection functions**:

$$\otimes : J_R(A^+) \times J_R(B^+) \to J_R(A^+ \times B^+)$$

- To interpret **induction** it is sufficient to interpret:

$$\forall i < n \neg\neg A(i) \to \neg\neg\forall i < n\, A(i)$$

which can be done via the **finite product of selection functions**:

$$\bigotimes_n : \Pi_{i<n} J_R(A^+) \to J_R(\Pi_{i<n} A^+)$$

Martín Escardó, P. O. & Thomas Powell, System T and the products of selection functions, 2011

Countable choice

- Countable choice $\mathbf{AC}_0$ is the axiom schema

$$\forall n^{\mathbb{N}} \exists x^{\tau} A_n(x) \to \exists \alpha^{\mathbb{N} \to \tau} \forall n A_n(\alpha(n))$$

- To interpret $\mathbf{AC}_0$ it is sufficient to interpret **DNS**

$$\forall n^{\mathbb{N}} \neg\neg A(n) \to \neg\neg \forall n\, A(n)$$

using the **unbounded product of selection functions**

$$\bigotimes : \Pi_{n\,:\,\mathbb{N}}\, J_R(A_n^+) \to J_R(\Pi_{n\,:\,\mathbb{N}}\, A_n^+)$$

(shown to be equivalent to bar recursion)

Martín Escardó & P. O., Bar recursion and the product of selection functions, 2015

Summary

- Proofs often carry more information than what is stated in theorem
- **Functional interpretations** view formulas as triples

$$A \quad \mapsto \quad (A^+, A^-, |A| \subseteq A^+ \times A^-)$$

- These can be seen as **games** between two players
- **Implication** is interpreted as two-way maps between games
- Two degrees of freedom: **atomic formulas** & **contraction**
- **Negative translations** are simplifications of Kolmogorov translation
- Realizers for double negated formulas are **selection functions**
- The **product of selection functions** (classically) interprets countable choice and comprehension *(equivalent to bar recursion)*

Thank You!