Uniform Realizability Interpretations

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Continuity, Computability, Constructivity - From Logic to Algorithms

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Plan

- Realizability Interpretations
  (background)
- Uniform Interpretations of Quantifiers
  (a bit of history...)
- A Uniform Realizability Interpretation
  (parametrised by a base interpretation)
- Some Base Interpretations
  (examples of base interpretations)

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Realizability Interpretations

- Interpret a **formula** A as a set of (computable) functions A^r
- Interpret **proofs** of A as elements of A^r
- Key idea: "Skolemize" A as $\exists \vec{x} A_{ef}(\vec{x})$

$$A^r \;:\equiv \{\vec{t} \mid A_{ef}(\vec{t})\}$$

- E.g. if $A \equiv \forall n \exists p \geq n \, \mathrm{Prime}(p)$ then

$$A^r \;:\equiv \{t \mid \forall n(t(n) \geq n \wedge \mathrm{Prime}(t(n)))\}$$

- So, from a proof that there are infinitely many primes we can extract a program that computes arbitrarily large primes

Realizability Interpretations

- Key idea: "Skolemize" A as $\exists \vec{x} A_{ef}(\vec{x})$

$$A^r \ :\equiv \{\vec{t} \mid A_{ef}(\vec{t})\}$$

Definition (Realizability Interpretation).

Define $\boldsymbol{a}$ r A by induction on the formula A:

$$\begin{aligned}
\langle\,\rangle \text{ r } P(\vec{x}) \quad &:\equiv \quad P(\vec{x}) \\
\boldsymbol{a}, \boldsymbol{b} \text{ r } A \wedge B \quad &:\equiv \quad (\boldsymbol{a} \text{ r } A) \wedge (\boldsymbol{b} \text{ r } B) \\
\boldsymbol{f} \text{ r } A \to B \quad &:\equiv \quad \forall \boldsymbol{a}((\boldsymbol{a} \text{ r } A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \boldsymbol{f} \bullet \boldsymbol{a} \text{ r } B)) \\
k, \boldsymbol{a} \text{ r } \exists n^{\mathbb{N}} A \quad &:\equiv \quad \boldsymbol{a} \text{ r } A[k/n] \\
\boldsymbol{f} \text{ r } \forall n^{\mathbb{N}} A \quad &:\equiv \quad \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ r } A)
\end{aligned}$$

- So, $A^r \equiv \{\boldsymbol{a} \mid \boldsymbol{a} \text{ r } A\}$, i.e. $A_{ef}(\vec{t})$ can be defined inductively

Realizability Interpretations

- Skolemization relies on **AC**

$$\forall x^\rho \exists y^\tau A(x,y) \to \exists f^{\rho\to\tau} \forall x^\rho A(x, f(x))$$

- In general we do not have

$$\forall x^\rho \exists y^\tau A(x,y) \to \exists y^\tau \forall x^\rho A(x,y)$$

- But, sometimes we do!
  - **Pointwise continuity** implies **uniform continuity**

    $\forall f \exists n \forall g \ldots \to \exists n \forall f, g \ldots$

  - **Bounded collection** (when $A(n,m)$ monotone in m)

    $\forall n \leq k \exists m\, A(n,m) \to \exists m \forall n \leq k\, A(n,m)$

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Uniform Interpretations of Quantifiers

Typed lambda-calculus in classical Zermelo-Frænkel set theory

2001

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In this paper, we develop a system of typed lambda-calculus for the Zermelo-Frænkel set theory, in the framework of classical logic. The first, and the simplest system of typed lambda-calculus is the *system of simple types*, which uses the intuitionistic propositional calculus, with the only connective $\to$. It is very important, because the well known Curry-

The definition is given by induction on F :

$$|F \to G| = (|F| \to |G|) \ ; \ |\forall x\, F| = \bigcap_a |F[a/x]|.$$

Therefore :

$t \Vdash (F \to G)$ is the formula $(\forall u \in \Lambda)(u \Vdash F \to tu \Vdash G)$;
$t \Vdash \forall x\, F$ is the formula $\forall x(t \Vdash F)$.

Uniform Interpretations of Quantifiers

UNIFORM HEYTING ARITHMETIC

2003

ULRICH BERGER

Dedicated to Helmut Schwichtenberg on his 60th Birthday

Abstract. We present an extension of Heyting Arithmetic in finite types called *Uniform Heyting Arithmetic* ($\mathsf{HA^u}$) that allows for the extraction of optimized programs from constructive and classical proofs. The system $\mathsf{HA^u}$ has two sorts of first-order quantifiers: ordinary quantifiers governed by the usual rules, and uniform quantifiers subject to stronger variable conditions expressing roughly that the quantified object is not computationally used in the proof. We combine a Kripke-style Friedman/Dragalin translation which is inspired by work of Coquand and Hofmann and a variant of the refined A-translation due to Buchholz, Schwichtenberg and the author to extract programs from a rather large class of classical first-order proofs while keeping explicit control over the levels of recursion and the decision procedures for predicates used in the extracted program.

→

$$r\ \mathbf{mr}\ \exists x^\rho\, A = \begin{cases} \mathsf{p}_1(r)\ \mathbf{mr}\ A[\mathsf{p}_0(r)/x] & \text{if } A \text{ is non-Harrop} \\ \epsilon\ \mathbf{mr}\ A[r/x] & \text{if } A \text{ is Harrop} \end{cases}$$

$$r\ \mathbf{mr}\ \mathsf{Q}A = \mathsf{Q}(r\ \mathbf{mr}\ A) \text{ where } \mathsf{Q} \in \{\{\forall x\}, \{\exists x\}\}$$

Uniform Interpretations of Quantifiers

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2003

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COMPUTING WITH INFINITE OBJECTS: THE GRAY CODE CASE

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Program extraction is performed via a 'uniform' realisability interpretation (Section 4.4). Uniformity concerns the interpretation of quantifiers: A formula $(\forall x)\, A(x)$ is realised uniformly by one object a that realises $A(x)$ for all x, so a may not depend on x. Dually, a formula $(\exists x)\, A(x)$ is realised uniformly by one object a that realises $A(x)$ for some x, so a does not contain a witness for x. Expressions (formulas, predicates, operators) that contain

Uniform Interpretations of Quantifiers

A functional interpretation for nonstandard arithmetic

2012

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ABSTRACT

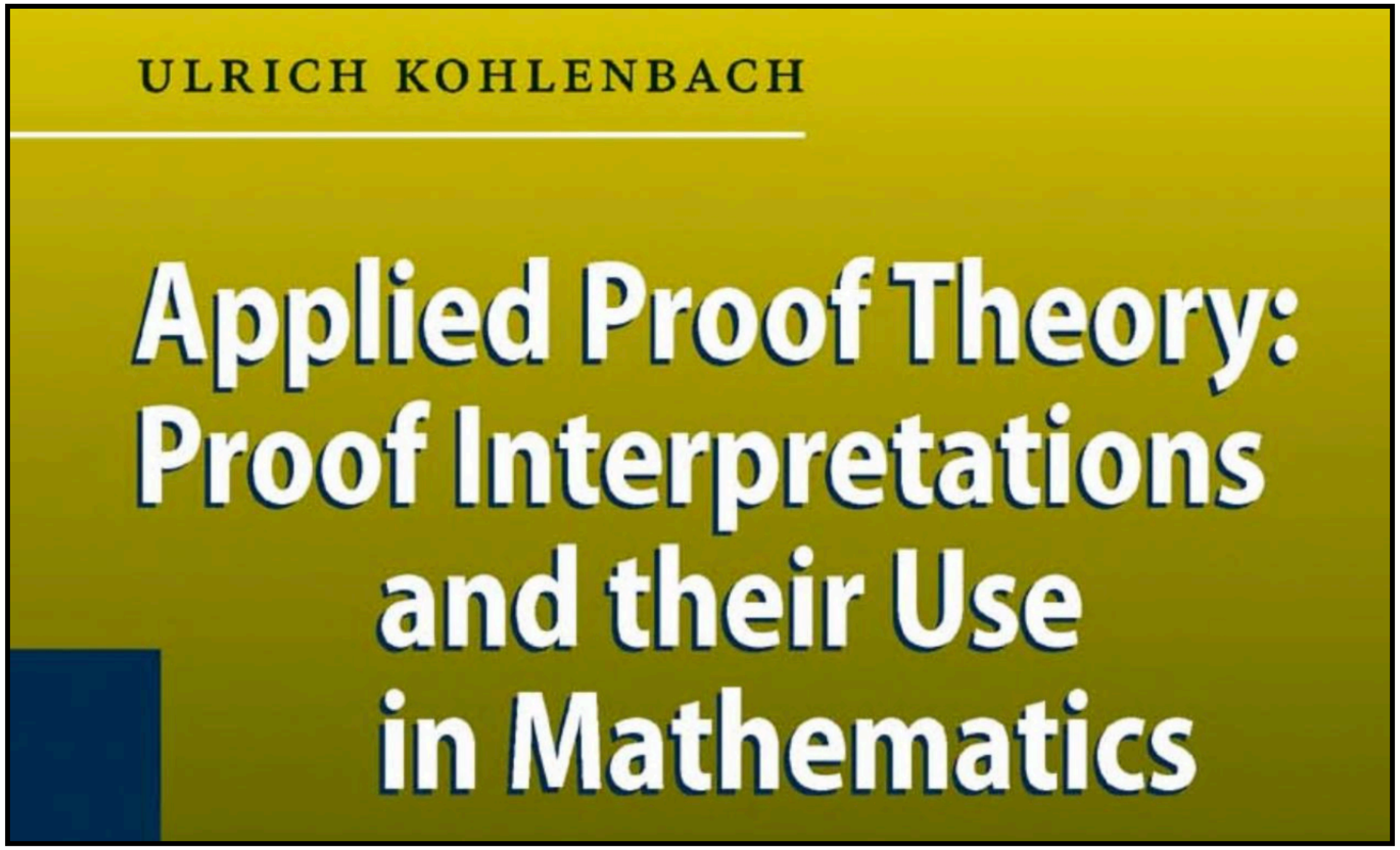
We introduce constructive and classical systems for nonstandard arithmetic and show how variants of the functional interpretations due to Gödel and Shoenfield can be used to rewrite proofs performed in these systems into standard ones. These functional interpretations show in particular that our nonstandard systems are conservative extensions of E-HA$^\omega$ and E-PA$^\omega$, strengthening earlier results by Moerdijk and Palmgren, and Avigad and Helzner. We will also indicate how our rewriting algorithm can be used for term extraction purposes. To conclude the paper, we will point out some open problems and directions for future research, including some initial results on saturation principles.

$$
\begin{aligned}
\underline{s}\ \mathrm{hr}\,\exists x\,\Phi(x) &:\equiv \exists x\,\big(\underline{s}\,\mathrm{hr}\,\Phi(x)\big), \\
\underline{s}\ \mathrm{hr}\,\forall x\,\Phi(x) &:\equiv \forall x\,\big(\underline{s}\,\mathrm{hr}\,\Phi(x)\big), \\
s,\underline{t}\ \mathrm{hr}\,\exists^{\mathrm{st}} x\,\Phi(x) &:\equiv \exists s' \in s\,\big(\underline{t}\,\mathrm{hr}\,\Phi(s')\big), \\
\underline{s}\ \mathrm{hr}\,\forall^{\mathrm{st}} x\,\Phi(x) &:\equiv \forall^{\mathrm{st}} x\,\big(\underline{s}[x]\,\mathrm{hr}\,\Phi(x)\big).
\end{aligned}
$$

(internal)

(external)

Uniformity in Proof Mining



This book gives an introduction to so-called proof interpretations, more specifically various forms of realizability and functional interpretations, and their use in mathematics. Whereas earlier treatments of these techniques (e.g. [362, 264, 121, 365, 7]) emphasize foundational and logical issues the focus of this book is on applications of the methods to extract new effective information such as computable uniform bounds from given (typically ineffective) proofs. This line of research, which has its roots in G. Kreisel's pioneering work on 'unwinding of proofs' from the 50's, has

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Heyting Arithmetic

Definition (Heyting arithmetic).

Assume $\mathbf{HA}$ formalised with three predicate symbols

- Falsity $\bot$ — nullary
- Natural number $\mathbb{N}(n)$ — unary
- Equality $n = m$ — binary

Notation.

$\forall n^{\mathbb{N}} A(n)$ is an abbreviation for $\forall n(\mathbb{N}(n) \to A(n))$

$\exists n^{\mathbb{N}} A(n)$ is an abbreviation for $\exists n(\mathbb{N}(n) \wedge A(n))$

A Uniform Realizability Interpretation

Definition (Base Interpretation).

Associate to each n-ary predicate symbol P an $(n+m)$-ary relation

$$\vec{x} \lhd_P \boldsymbol{a}$$

between individuals and P-realizers (or P-bounds).

Examples.

For the unary predicate $\mathbb{N}(n)$ we could take $n \lhd_{\mathbb{N}} \cdot$ to be:

$$\begin{aligned} n \lhd_{\mathbb{N}} m \;&:\equiv\; n = m && \text{(precise)} \\ n \lhd_{\mathbb{N}} m \;&:\equiv\; n \leq m && \text{(bounded)} \\ n \lhd_{\mathbb{N}} S \;&:\equiv\; n \in S && \text{(Herbrand)} \\ n \lhd_{\mathbb{N}} \langle\,\rangle \;&:\equiv\; \text{true} && \text{(uniform)} \end{aligned}$$

A Uniform Realizability Interpretation

Definition (Uniform Realizability Interpretation).

Given a base interpretation. Let:

$$
\begin{aligned}
\boldsymbol{a} \text{ ur } P(\vec{x}) &:\equiv \vec{x} \lhd_P \boldsymbol{a} \\
\boldsymbol{a}, \boldsymbol{b} \text{ ur } A \wedge B &:\equiv (\boldsymbol{a} \text{ ur } A) \wedge (\boldsymbol{b} \text{ ur } B) \\
\boldsymbol{f} \text{ ur } A \to B &:\equiv \forall \boldsymbol{a}((\boldsymbol{a} \text{ ur } A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge\, \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } B)) \\
\boldsymbol{a} \text{ ur } \exists x A &:\equiv \exists x (\boldsymbol{a} \text{ ur } A) \\
\boldsymbol{a} \text{ ur } \forall x A &:\equiv \forall x (\boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

It follows that...

$$
\begin{aligned}
\boldsymbol{a}, \boldsymbol{b} \text{ ur } \exists n^{\mathbb{N}} A &:\equiv \exists n \lhd_{\mathbb{N}} \boldsymbol{a}\ (\boldsymbol{b} \text{ ur } A) \\
\boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &:\equiv \forall \boldsymbol{a} \forall n \lhd_{\mathbb{N}} \boldsymbol{a}\ (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge\, \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

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Kleene Realizability

Definition (Kleene Base Interpretation).

Let:

$$
\begin{aligned}
\langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\
n \lhd_{\mathbb{N}} m &:\equiv n = m \\
(n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m
\end{aligned}
$$

It follows that...

$$
\begin{aligned}
n, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(n) \\
\boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n(\{\boldsymbol{a}\}(n) \downarrow \wedge \{\boldsymbol{a}\}(n) \text{ ur } A)
\end{aligned}
$$

Kreisel Modified Realizability

Definition (Kreisel Base Interpretation).

Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\ n \lhd_{\mathbb{N}} m^{\mathbb{N}} &:\equiv n = m \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

It follows that...

$$\begin{aligned} n^{\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(n) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ ur } A) \end{aligned}$$

Herbrand Realizability

Definition (Herbrand Base Interpretation).

Assume an extra unary predicate $\mathrm{std}(n)$ (for n is a **standard number**). Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\ n \lhd_{\mathbb{N}} \langle\,\rangle &:\equiv \text{true} \\ n \lhd_{\mathrm{std}} S &:\equiv n \in S \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \exists n^{\mathbb{N}} (\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A(n) &\Leftrightarrow \forall n^{\mathbb{N}} (\boldsymbol{a} \text{ ur } A(n)) \\ S^{\mathbb{N}^*}, \boldsymbol{a} \text{ ur } \exists n^{\mathrm{std}} A(n) &\Leftrightarrow \exists n \in S\, (\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathrm{std}} A &\Leftrightarrow \forall S\, \forall n \in S\, (\boldsymbol{f}(S) \text{ ur } A) \end{aligned}$$

Classical Modified Realizability

Definition (Classical Base Interpretation).

Fix unary atomic predicate $P_\perp(n)$. Let:

$$\begin{aligned} \langle\,\rangle \lhd_\perp n &:\equiv P_\perp(n) \\ n \lhd_{\mathbb{N}} m^{\mathbb{N}} &:\equiv n = m \\ (n,m) \lhd_= \langle\,\rangle &:\equiv n = m \end{aligned}$$

Remarks.

- Combination of modified realizability and Friedman's A-translation
- We are then able to realize $\neg\neg\exists n^{\mathbb{N}} P_\perp(n) \to \exists n^{\mathbb{N}} P_\perp(n)$
- Similar to Krivine's (classical) realizability

Aschieri-Berardi Learning Realizability

Definition (Aschieri-Berardi Base Interpretation).

Assume a set of states $\mathbf{S}$. Parametrised by an $s \in \mathbf{S}$, let:

$$\begin{aligned} \langle\,\rangle \lhd_{\perp} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) \neq s \\ n \lhd_{\mathbb{N}} \alpha^{\mathbf{S}\to\mathbb{N}} &:\equiv n = \alpha(s) \\ (n, m) \lhd_{=} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) = s \to n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \alpha^{\mathbf{S}\to\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(\alpha(s)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n^{\mathbb{N}}(\boldsymbol{f}(n) \text{ ur } A) \end{aligned}$$

The uniform functional interpretation with informative types

F. Ferreira and P. Oliva

Definition 2.3 ($\mathcal{U}$-interpretation of $\mathcal{L}_{\mathbf{S}}$ into $\mathcal{L}_{\mathbf{T}}$). *Let be given a base interpretation of $\mathcal{L}_{\mathbf{S}}$ into $\mathcal{L}_{\mathbf{T}}$. For each formula A of $\mathcal{L}_{\mathbf{S}}$, we define its $\mathcal{U}$-*interpretation $\langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}}$ *into $\mathcal{L}_{\mathbf{T}}$. The definition is by induction on the logical structure of A.*

For atomic formulas $R(t_1, \ldots, t_n)$, its $\mathcal{U}$-interpretation is defined as the given information relation $\langle R(t_1, \ldots, t_n) \rangle^{\boldsymbol{c}}_{\boldsymbol{d}}$. For $\bot$ we define

$$\langle \bot \rangle \quad :\equiv \quad \bot.$$

Assuming that A and B have $\mathcal{U}$-interpretations $\langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\langle B \rangle^{\boldsymbol{c}}_{\boldsymbol{d}}$, respectively, we define:

$$\begin{array}{lcl}
\langle A \wedge B \rangle^{\boldsymbol{a},\boldsymbol{c}}_{\boldsymbol{b},\boldsymbol{d}} & :\equiv & \langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} \wedge \langle B \rangle^{\boldsymbol{c}}_{\boldsymbol{d}} \\
\langle A \to B \rangle^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{a},\boldsymbol{d}} & :\equiv & \forall \boldsymbol{b} \in \boldsymbol{gad} \langle A \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} \to \langle B \rangle^{\boldsymbol{fa}}_{\boldsymbol{d}} \\
\langle \forall x^\sigma A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} & :\equiv & \forall x^\sigma \langle A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{b}} \\
\langle \exists x^\sigma A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{B}} & :\equiv & \exists x^\sigma \forall \boldsymbol{b} \in \boldsymbol{B} \, \langle A(x) \rangle^{\boldsymbol{a}}_{\boldsymbol{b}}.
\end{array}$$

Summary

- Quantifiers are "naturally" **uniform** (non-computational)
- Qualified quantifications (e.g. $\exists n^{\mathbb{N}} A(n)$) carry computational content because of the qualifying predicate $\mathbb{N}(n)$
- Currently working with Fernando Ferreira on uniform functional interpretations:
  - New interpretations of function spaces $\rho \to \tau$
  - Functional interpretation of extensionality
  - Systematic treatment of bounded (uniform) quantifiers