On Uniform Proof Interpretations

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Plan

- Realizability Interpretations
  (background)
- Uniform Interpretations of Quantifiers
  (a bit of history...)
- A Uniform Realizability Interpretation
  (parametrised by a base interpretation)
- Some Base Interpretations
  (examples of base interpretations)

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Realizability Interpretations

- Interpret a **formula** A as a set of (computable) functions A^r
- Interpret **proofs** of A as elements of A^r
- Key idea: "Skolemize" A as $\exists \vec{x} A_{ef}(\vec{x})$

$$A^r \;:\equiv \{\vec{t} \mid A_{ef}(\vec{t})\}$$

- E.g. if $A \equiv \forall n \exists p \geq n \, \mathrm{Prime}(p)$ then

$$A^r \;:\equiv \{t \mid \forall n(t(n) \geq n \wedge \mathrm{Prime}(t(n)))\}$$

- So, from a proof that there are infinitely many primes we can extract a program that computes arbitrarily large primes

Realizability Interpretations

- Key idea: "Skolemize" A as $\exists \vec{x} A_{ef}(\vec{x})$

$$A^r \;:\equiv\; \{\vec{t} \mid A_{ef}(\vec{t})\}$$

Definition (Realizability Interpretation).

Define $\boldsymbol{a}$ r A by induction on the formula A:

$$\begin{aligned}
\langle\,\rangle \text{ r } P(\vec{x}) &:\equiv P(\vec{x}) \\
\boldsymbol{a}, \boldsymbol{b} \text{ r } A \wedge B &:\equiv (\boldsymbol{a} \text{ r } A) \wedge (\boldsymbol{b} \text{ r } B) \\
\boldsymbol{f} \text{ r } A \to B &:\equiv \forall \boldsymbol{a}((\boldsymbol{a} \text{ r } A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \boldsymbol{f} \bullet \boldsymbol{a} \text{ r } B)) \\
k, \boldsymbol{a} \text{ r } \exists n^{\mathbb{N}} A &:\equiv \boldsymbol{a} \text{ r } A[k/n] \\
\boldsymbol{f} \text{ r } \forall n^{\mathbb{N}} A &:\equiv \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ r } A)
\end{aligned}$$

- So, $A^r \equiv \{\boldsymbol{a} \mid \boldsymbol{a} \text{ r } A\}$, i.e. $A_{ef}(\vec{t})$ can be defined inductively

Realizability Interpretations

- Skolemization relies on **AC**

$$\forall x^\rho \exists y^\tau A(x, y) \to \exists f^{\rho\to\tau} \forall x^\rho A(x, f(x))$$

- <u>In general</u> we do not have

$$\forall x^\rho \exists y^\tau A(x, y) \to \exists y^\tau \forall x^\rho A(x, y)$$

- But, <u>sometimes we do!</u>
  - **Pointwise continuity** implies **uniform continuity**

    $\forall f \exists n \forall g \ldots \to \exists n \forall f, g \ldots$

  - **Bounded collection** (when $A(n, m)$ monotone in m)

    $\forall n \leq k \exists m\, A(n, m) \to \exists m \forall n \leq k\, A(n, m)$

The Effective Topos

J.M.E. Hyland
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Corollary 15.2. *The "Uniformity Principle"*

$$\forall\phi[\forall X \in \mathcal{P}(\mathbb{N}).\exists n \in \mathbb{N}.\phi(X,n) \to \exists n \in \mathbb{N}.\forall X \in \mathcal{P}(\mathbb{N}).\phi(X,n)]$$

holds in $\mathcal{E}\!f\!f$.

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Uniform Interpretations of Quantifiers

Typed lambda-calculus in classical Zermelo-Frænkel set theory

2001

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In this paper, we develop a system of typed lambda-calculus for the Zermelo-Frænkel set theory, in the framework of classical logic. The first, and the simplest system of typed lambda-calculus is the *system of simple types*, which uses the intuitionistic propositional calculus, with the only connective $\rightarrow$. It is very important, because the well known Curry-

The definition is given by induction on F :

$$|F \rightarrow G| = (|F| \rightarrow |G|) \ ; \ |\forall x\, F| = \bigcap_a |F[a/x]|.$$

Therefore :

$t \Vdash (F \rightarrow G)$ is the formula $(\forall u \in \Lambda)(u \Vdash F \rightarrow tu \Vdash G)$;

$t \Vdash \forall x\, F$ is the formula $\forall x(t \Vdash F)$.

Uniform Interpretations of Quantifiers

2003

UNIFORM HEYTING ARITHMETIC

ULRICH BERGER

Dedicated to Helmut Schwichtenberg on his 60th Birthday

Abstract. We present an extension of Heyting Arithmetic in finite types called *Uniform Heyting Arithmetic* (HA^{u}) that allows for the extraction of optimized programs from constructive and classical proofs. The system HA^{u} has two sorts of first-order quantifiers: ordinary quantifiers governed by the usual rules, and uniform quantifiers subject to stronger variable conditions expressing roughly that the quantified object is not computationally used in the proof. We combine a Kripke-style Friedman/Dragalin translation which is inspired by work of Coquand and Hofmann and a variant of the refined A-translation due to Buchholz, Schwichtenberg and the author to extract programs from a rather large class of classical first-order proofs while keeping explicit control over the levels of recursion and the decision procedures for predicates used in the extracted program.

→

$$r\,\mathbf{mr}\,\exists x^\rho\,A = \begin{cases} \mathsf{p}_1(r)\,\mathbf{mr}\,A[\mathsf{p}_0(r)/x] & \text{if } A \text{ is non-Harrop} \\ \epsilon\,\mathbf{mr}\,A[r/x] & \text{if } A \text{ is Harrop} \end{cases}$$

$$r\,\mathbf{mr}\,\mathsf{Q}A = \mathsf{Q}(r\,\mathbf{mr}\,A) \text{ where } \mathsf{Q} \in \{\{\forall x\}, \{\exists x\}\}$$

Uniform Interpretations of Quantifiers

A functional interpretation for nonstandard arithmetic

2012

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ABSTRACT

We introduce constructive and classical systems for nonstandard arithmetic and show how variants of the functional interpretations due to Gödel and Shoenfield can be used to rewrite proofs performed in these systems into standard ones. These functional interpretations show in particular that our nonstandard systems are conservative extensions of E-HA$^{\omega}$ and E-PA$^{\omega}$, strengthening earlier results by Moerdijk and Palmgren, and Avigad and Helzner. We will also indicate how our rewriting algorithm can be used for term extraction purposes. To conclude the paper, we will point out some open problems and directions for future research, including some initial results on saturation principles.

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$$
\begin{aligned}
\underline{s}\ \mathrm{hr}\, \exists x\, \Phi(x) &:\equiv \exists x\, \big(\underline{s}\, \mathrm{hr}\, \Phi(x)\big), \\
\underline{s}\ \mathrm{hr}\, \forall x\, \Phi(x) &:\equiv \forall x\, \big(\underline{s}\, \mathrm{hr}\, \Phi(x)\big), \\
s, \underline{t}\ \mathrm{hr}\, \exists^{\mathrm{st}} x\, \Phi(x) &:\equiv \exists s' \in s\, \big(\underline{t}\, \mathrm{hr}\, \Phi(s')\big), \\
\underline{s}\ \mathrm{hr}\, \forall^{\mathrm{st}} x\, \Phi(x) &:\equiv \forall^{\mathrm{st}} x\, \big(\underline{s}[x]\, \mathrm{hr}\, \Phi(x)\big).
\end{aligned}
$$

(internal)

(external)

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Heyting Arithmetic

Definition (Heyting arithmetic).

Assume **HA** formalised with three predicate symbols

- Falsity $\bot$ — nullary
- Natural number $\mathbb{N}(n)$ — unary
- Equality $n = m$ — binary

Notation.

$\forall n^{\mathbb{N}} A(n)$ is an abbreviation for $\forall n(\mathbb{N}(n) \to A(n))$

$\exists n^{\mathbb{N}} A(n)$ is an abbreviation for $\exists n(\mathbb{N}(n) \wedge A(n))$

A Uniform Realizability Interpretation

Definition (Base Interpretation).

Associate to each n-ary predicate symbol P an $(n+m)$-ary relation

$$\vec{x} \lhd_P \boldsymbol{a}$$

between individuals and P-realizers (or P-bounds).

Examples.

For the unary predicate $\mathbb{N}(n)$ we could take $n \lhd_{\mathbb{N}} \cdot$ to be:

$$n \lhd_{\mathbb{N}} m \;:\equiv\; n = m \qquad \text{(precise)}$$

$$n \lhd_{\mathbb{N}} m \;:\equiv\; n \leq m \qquad \text{(bounded)}$$

$$n \lhd_{\mathbb{N}} S \;:\equiv\; n \in S \qquad \text{(Herbrand)}$$

$$n \lhd_{\mathbb{N}} \langle\,\rangle \;:\equiv\; \text{true} \qquad \text{(uniform)}$$

A Uniform Realizability Interpretation

Definition (Uniform Realizability Interpretation).

Given a base interpretation. Let:

$$
\begin{aligned}
\boldsymbol{a} \text{ ur } P(\vec{x}) &:\equiv \vec{x} \lhd_P \boldsymbol{a} \\
\boldsymbol{a},\boldsymbol{b} \text{ ur } A \land B &:\equiv (\boldsymbol{a} \text{ ur } A) \land (\boldsymbol{b} \text{ ur } B) \\
\boldsymbol{f} \text{ ur } A \to B &:\equiv \forall \boldsymbol{a}((\boldsymbol{a} \text{ ur } A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \land \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } B)) \\
\boldsymbol{a} \text{ ur } \exists x A &:\equiv \exists x (\boldsymbol{a} \text{ ur } A) \\
\boldsymbol{a} \text{ ur } \forall x A &:\equiv \forall x (\boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

It follows that...

$$
\begin{aligned}
\boldsymbol{a},\boldsymbol{b} \text{ ur } \exists n^{\mathbb{N}} A &:\equiv \exists n \lhd_{\mathbb{N}} \boldsymbol{a}\, (\boldsymbol{b} \text{ ur } A) \\
\boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &:\equiv \forall \boldsymbol{a} \forall n \lhd_{\mathbb{N}} \boldsymbol{a}\, (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \land \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

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Kleene Realizability

Definition (Kleene Base Interpretation).

Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &\quad :\equiv \quad \bot \\ n \lhd_{\mathbb{N}} m &\quad :\equiv \quad n = m \\ (n, m) \lhd_{=} \langle\,\rangle &\quad :\equiv \quad n = m \end{aligned}$$

It follows that...

$$\begin{aligned} n, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\quad \Leftrightarrow \quad \boldsymbol{a} \text{ ur } A(n) \\ \boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A &\quad \Leftrightarrow \quad \forall n(\{\boldsymbol{a}\}(n)\downarrow \wedge \{\boldsymbol{a}\}(n) \text{ ur } A) \end{aligned}$$

Kreisel Modified Realizability

Definition (Kreisel Base Interpretation).

Let:

$$
\begin{aligned}
\langle\,\rangle \lhd_{\perp} \langle\,\rangle \quad &:\equiv \quad \perp \\
n \lhd_{\mathbb{N}} m^{\mathbb{N}} \quad &:\equiv \quad n = m \\
(n, m) \lhd_{=} \langle\,\rangle \quad &:\equiv \quad n = m
\end{aligned}
$$

It follows that...

$$
\begin{aligned}
n^{\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) \quad &\Leftrightarrow \quad \boldsymbol{a} \text{ ur } A(n) \\
\boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A \quad &\Leftrightarrow \quad \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ ur } A)
\end{aligned}
$$

Herbrand Realizability

Definition (Herbrand Base Interpretation).

Assume an extra unary predicate $\mathrm{std}(n)$ (for n is a **standard number**). Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\ n \lhd_{\mathbb{N}} \langle\,\rangle &:\equiv \text{true} \\ n \lhd_{\mathrm{std}} S &:\equiv n \in S \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \exists n^{\mathbb{N}} (\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A(n) &\Leftrightarrow \forall n^{\mathbb{N}} (\boldsymbol{a} \text{ ur } A(n)) \\ S^{\mathbb{N}^*}, \boldsymbol{a} \text{ ur } \exists n^{\mathrm{std}} A(n) &\Leftrightarrow \exists n \in S\ (\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathrm{std}} A &\Leftrightarrow \forall S\, \forall n \in S\ (\boldsymbol{f}(S) \text{ ur } A) \end{aligned}$$

Classical Modified Realizability

Definition (Classical Base Interpretation).

Fix unary atomic predicate $P_\perp(n)$. Let:

$$\begin{aligned} \langle\,\rangle \lhd_\perp n &\;:\equiv\; P_\perp(n) \\ n \lhd_{\mathbb{N}} m^{\mathbb{N}} &\;:\equiv\; n = m \\ (n, m) \lhd_= \langle\,\rangle &\;:\equiv\; n = m \end{aligned}$$

Remarks.

- Combination of modified realizability and Friedman's A-translation
- We are then able to realize $\neg\neg\exists n^{\mathbb{N}} P_\perp(n) \to \exists n^{\mathbb{N}} P_\perp(n)$
- Similar to Krivine's (classical) realizability

Aschieri-Berardi Learning Realizability

Definition (Aschieri-Berardi Base Interpretation).

Assume a set of states $\mathbf{S}$. Parametrised by an $s \in \mathbf{S}$, let:

$$\begin{aligned} \langle\,\rangle \lhd_{\perp} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) \neq s \\ n \lhd_{\mathbb{N}} \alpha^{\mathbf{S}\to\mathbb{N}} &:\equiv n = \alpha(s) \\ (n, m) \lhd_{=} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) = s \to n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \alpha^{\mathbf{S}\to\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(\alpha(s)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n^{\mathbb{N}} (\boldsymbol{f}(n) \text{ ur } A) \end{aligned}$$

Summary

- Quantifiers are "naturally" **uniform** (non-computational)
- Qualified quantifications (e.g. $\exists n^{\mathbb{N}} A(n)$) carry computational content because of the qualifying predicate $\mathbb{N}(n)$
- Currently working with Fernando Ferreira on uniform functional interpretations:
  - New interpretations of function spaces $\rho \to \tau$
  - Functional interpretation of extensionality
  - Systematic treatment of bounded (uniform) quantifiers